PRACTICAL SHOP MATHEMATICS

VOLUME II—ADVANCED

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PRACTICAL SHOP MATHEMATICS

Volume II—Advanced

PREFACE TO THE THIRD EDITION

The rapid advances in the aircraft industries as well as continued advance and change in automobile designs make the theory of compound angles an increasingly important branch of mathematics for the draftsman, the mechanic, and the engineer.

Since it is easier for the layman to recognize the true conditions of a compound-angle problem drawn in a pictorial form, the authors have given methods by which the essential lines of any compound-angle problem may be converted into a pictorial form. Sufficient theory of orthographic projections, used in mechanical drawing, has been added to this text as a means of converting into a pictorial form the essential lines of a compound-angle problem that has been drawn in a mechanical drawing form.

To conform with the general shop practices, the authors have included in this edition of the text a normal amount of compound-angle problems drawn in a mechanical drawing form, some of which are completely worked out. These problems represent the five basic types of solid figures recommended by the authors. The insertion of these elementary problems will aid the student in becoming more familiar with the five basic types before entering into the study of the more difficult problems.

A more substantial treatise on compound-angular hole problems, together with a method of computing and checking the location of same with a ball-plug gage, has been added and will greatly aid the mechanic in overcoming compound-angular hole problem difficulties.

It is not necessary to have a rigid training in solid geometry in order to obtain a fair working knowledge of compound angles. A few of the necessary and most important proposi-

tions of solid geometry have been added and will suffice for the most difficult problems in compound angles.

The material and theory given in this entire text are suitable for modern aircraft and automotive industries and are presented in such a manner as to make each subject as comprehensible as possible.

Many original and simplified formulas, all of which are very useful to the layman, draftsman, and engineer, are included in this text. To be more proficient, every mechanic, draftsman, and engineer should have a working knowledge of the material and theory as stated in this text. This text should be a ready reference for all highly skilled mechanics, draftsmen, and engineers.

John H. Wolfe
Everett R. Phelps

PREFACE TO THE FIRST EDITION

For several years a course in shop mathematics has been taught under the guidance of John H. Wolfe at the Ford Apprentice School of the Ford Motor Company. The substance of this course was presented on loose-leaf printed sheets which were frequently revised in an effort to treat the material in the simplest and most understandable manner. The material in this book and in the preceding volume (Volume I) is the result of this careful revision and includes the work already developed in loose-leaf form, with a great deal of new and important material which has never before been presented.

To understand the work of this book, the pupil must possess a knowledge of fractions, decimals, geometry, and elementary plane trigonometry. If the student lacks this knowledge, the authors recommend the study of Volume I of this text, which deals with these subjects (and certain others as taper per foot, verniers, etc.) and their applications to practical shop problems.

The work on compound angles is very useful and is presented in book form for the first time. The method of attack and simplicity of approach as presented in this text will enable the student to solve many compound-angle problems for direct machine setups on jobs that formerly were machined only by the wasteful "cut-and-try" method. This second volume may well be used as the basis of a course to replace the usual high-school courses in plane trigonometry and solid geometry.

The value of this text in teaching the shop mathematics necessary to solve actual shop problems will be highly appreciated by anyone who has worked in this field. The exposition of the principles involved, the solution of many solved

practical problems, and the presentation of hundreds of problems for the student to solve (many of which are accompanied by hints for the solution) teach the student the general methods of deriving solutions which can be applied to all shop problems. Mr. Wolfe's fourteen years of machine-shop experience preceding his seventeen years of teaching shop mathematics have given him the proper background to present the many practical problems which originated in the factory tool rooms, die rooms, and drafting rooms.

One of the features of this book is the use of what the authors call the "variable system." Instead of all dimensions of a problem being given, one has been omitted and its value represented by a letter called the variable. Adjacent to the problem, or immediately after each exercise, six or seven values for the variable are given, all of which may be used for the omitted dimension. Thus the instructor can, by using the six given values for the variable, assign separate problems of the same type to six students, each of whom will obtain a different answer. This helps greatly in preventing students from comparing work and answers. The student's own abilities are consequently developed to a much fuller extent. Of course, if the instructor does not care to make use of this method, he may assign the same value of the variable to all members of the class. Whenever seven values for the variable are given, the answer for the seventh one accompanies the figure.

The explanations in this text are presented very completely so that a student or mechanic can profitably use the book for home study or as a reference source. The solutions accompanying many of the problems are also a great help to such students.

The authors wish to thank Mr. John W. Busman and Mr. William F. Mueller of the Ford Apprentice School Faculty for their assistance in proofreading.

JOHN H. WOLFE
EVERETT R. PHELPS

CONTENTS

USE OF THE VARIABLE SYSTEM

In all problems, with a very few exceptions, one number or dimension is represented by a letter which is called the variable. This letter or variable has six or seven different numerical values which may be substituted for it to complete the problem as stated. This makes six or seven similar problems, each of which has a different answer. These six or seven values of the variable are given in tabular form at the end of a group of problems. For problems which are stated diagrammatically, the six values of the variable are usually placed to the right of the diagram. A seventh value of the variable and the corresponding answer are usually placed directly under the diagram.

To illustrate the use of the variable system when the variables are given in tabular form, consider problem 1 on page 242 which reads: "When the gear makes G revolutions, how many teeth pass the pointer B?" On page 243 immediately following this group of problems is a table of variables which gives for G the values: 7.6841, 10.305, 11.742, 13.612, 14.735, 14.838. Thus for one student problem 1 becomes: "When the gear makes 7.6841 revolutions, how many teeth pass the pointer B?" For a second student, it will read: "When the gear makes 10.305 revolutions, how many teeth pass the pointer B?"; etc.

To illustrate the use of the variable system when variables are given to the right of the diagram, consider problem 1 at the bottom of page 28. β and ω are the angles to be computed and θ represents another angle, six values of which are given at the right of the figure. Any one of these values of θ may be used to complete the statement of the problem. Thus for one student the angle θ is 12°, for another 13°, etc.

Throughout the text, x, y, and z are used to represent

unknown distances, and any other letter of the English alphabet appearing in the problem is the variable. If the variable dimension is an angle, the Greek letter θ is generally used as the variable. Other Greek letters are used to represent the angular quantities to be computed.

A suggested plan for the use of the variable system in classrooms is presented in the chart below:

Name	No. of variable for 1st set of problems	No. of variable for 2d set of problems	No. of variable for 3d set of problems	Etc.
Brown, John	1	2	3	
Collins, Ray	2	3	4	
Grant, Peter.	3	4	5	
Hale, George	4	5	6	
Miller, Henry	5	6	1	
Smith, William	6	1	2	

The foregoing plan may be repeated for each group of six students.

The authors upon request will give further information regarding the use of the variable system.

PRACTICAL SHOP MATHEMATICS

CHAPTER I

SOLID TRIGONOMETRY OR COMPOUND ANGLES

This chapter introduces a new treatment of solid geometrical figures, which is to be referred to as **compound angles.** A special study of this material involves the relations of angles and sides of triangles that lie in different planes. This branch of mathematics is very useful in dealing with many practical problems that arise in the machining of surfaces and boring of holes at designated compound angles. They frequently appear in die sections, jig parts, fixture parts, the grinding of special cutting tools, the designing and construction of wings and bodies for airplanes, resultant of component forces given in two planes, the boring of compound-angular holes, and the milling of positive clutches and many other machine parts too numerous to mention.

Four or more planes, which are not parallel and which do not pass through the same point, form a solid figure. If only four planes are involved, the solid figure formed is a **triangular pyramid** as ABC-D of Fig. 1.

The planes intersect in points called **vertices** and in straight lines called the **edges** of the solid figures. Thus point A is a vertex and AB, AC, etc., are edges. The portion of a plane included between the edges is called a **face.** A solid figure may be shown graphically by drawing the lines that represent the edges.

The angles in the faces at a given vertex are called the **face angles** of that vertex. Thus angles BAC, BAD, and DAC are the face angles at the vertex A. Where a face angle is a right

angle, it is designated by a **small arc.** In Fig. 1 the three face angles at B are all right angles.

If sufficient data is given so that one face angle at one vertex can be determined in terms of the trigonometric functions of two or more face angles lying in adjacent planes, the angle to be determined is called a **compound angle,** and the problem is known as a **compound-angle problem.**

A **dihedral angle** is the opening between two intersecting planes. The **plane angle of a dihedral angle** is the angle

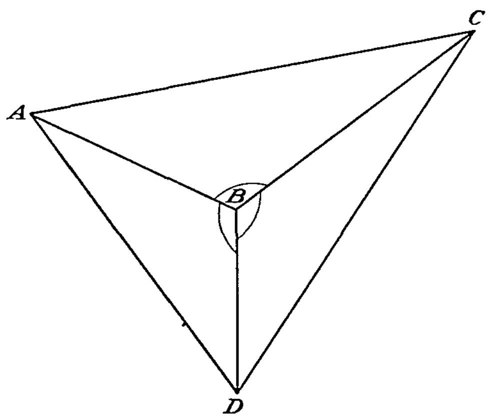


FIG. 1.

formed by two straight lines, drawn one in each face of the dihedral angle perpendicular to its edge at the same point. Thus the plane angle of the dihedral angle is always in a plane perpendicular to the line of intersection of the two given planes. This angle will be referred to as the **plane angle** of the edge formed by the two intersecting planes.

Example: The angle CDE in Fig. 2 is the plane angle of the edge AB.

The solution of compound-angle problems will be much more easily understood if the student will obtain plastic clay and make the solid figures that are to be discussed. The use of plastic clay in this manner is very important, as it will help the student to acquire the art of visualizing lines and planes

that have been drawn in space. The grasp of this type of visualization in the study of surfaces, angles, and edges of solid figures is an important factor and may be developed much earlier by the use of plastic-clay models cut out by the student. If a cross-sectional view is necessary, the model may also be cut to show the section formed.

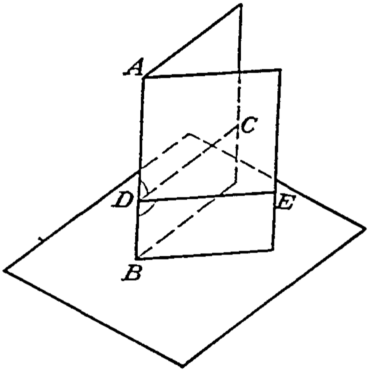


FIG. 2.

BASIC TYPES OF SOLID FIGURES

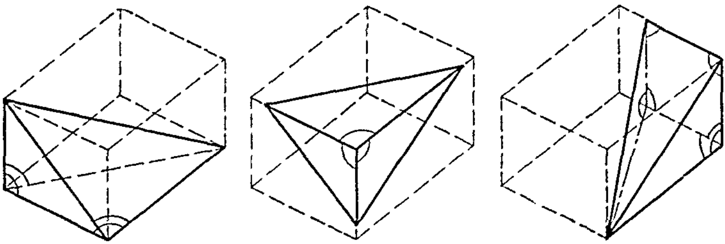
The method of attacking a compound-angle problem, generally known as the approach to a solution, is somewhat similar to the approach to a solution of a problem in plane trigonometry. In plane trigonometry, when sufficient data are given, a solution may be accomplished by skillfully arranging the construction lines into the **basic right triangles** or **oblique triangles**. In compound angles, when sufficient data are given, a solution may be accomplished by skillfully arranging the right triangles or oblique triangles into **basic types** of solid figures.

The authors have carefully selected five such basic types of solid figures, which, when properly applied, will serve as splendid means of computing any unknown angle or side of a solid figure when sufficient data are given.

Each type has a distinct shape, governed by the location of the right angles and the number of right triangles and oblique triangles that make up the basic types of solid figures. It is important, for rapid and successful coping with compound-

angle problems, that the shapes and the locations of the right angles of these five basic types be rigidly embedded in the minds of the students.

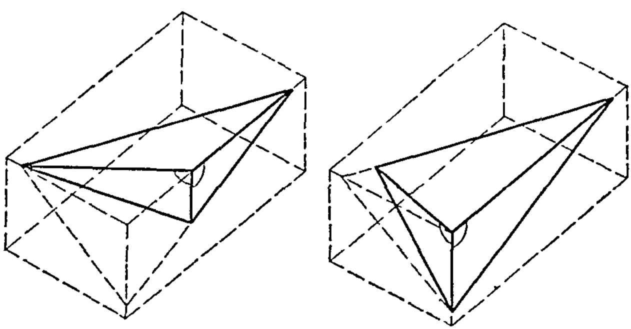
Very often the face angles, plane angles, or edges given in a solid figure are just sufficient to determine a solid, and it may



Type I Type II Type III

FIG. 3.

become necessary to solve for an unknown face angle, plane angle, or edge in terms of the given face angles, plane angles, or edges. Generally the practical compound-angle problem, with practice, can easily be associated with one of the five basic types or a combination of two or more of them.



Type IV Type V

FIG. 4.

In industry the part to be machined is shown on blueprint in a mechanical drawing form. Mechanical drawings are based on the principles of projecting the shape of the object in three conventional planes, known as the horizontal, vertical,

and profile planes. If the part to be machined contains one or more diagonal surfaces or a compound-angular hole, this is indicated by projecting the intersection lines of the diagonal surfaces or the projection lines of the compound-angular hole into these planes. Therefore, to be consistent with the rules of mechanical drawing the five chosen basic types, as shown in Figs. 3 and 4, have been inserted in a rectangular parallelepiped containing these three conventional planes. The insertion of them in a rectangular parallelepiped also serves as an aid to the student in obtaining a clear and proper conception of each of the five basic types.

As the student progresses in the study of compound angles, he will notice that each of the five basic types can be divided into other basic Type I solid figures. This feature is similar to the right triangle and the oblique triangle in plane geometry; they, too, can be divided into other right triangles. Therefore, in the pursuit of a solution the student should firmly have in mind the basic Type I solid figure, which is the fundamental type and cannot be subdivided into any other simpler form.

TRIGONOMETRIC FUNCTIONS IN TERMS OF THE UNITY METHOD

The unity method of defining the trigonometric functions is better adapted to the solution of compound-angle problems than the usual ratio method. In Volume I of "Practical Shop Mathematics," the trigonometric functions are defined by the unity method and for convenience are restated at this time.

In Fig. 5, sin α is numerically equal to the length of the side opposite (HI) when the hypotenuse (GI) is unity. Likewise, cos α is numerically equal to the length of the side adjacent (GH) when the hypotenuse (GI) is unity.

In Fig. 6, tan α is numerically equal to the length of the side opposite (KL) when the side adjacent (JK) is unity. Likewise, sec α is numerically equal to the length of the hypotenuse (JL) when the side adjacent (JK) is unity.

In Fig. 7, cot α is numerically equal to the length of the

side adjacent (MN) when the side opposite (NO) is unity. Likewise, csc α is numerically equal to the length of the hypotenuse (MO) when the side opposite (NO) is unity.

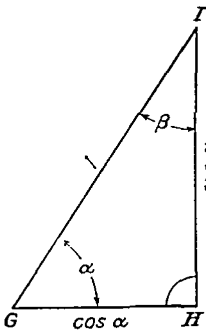


FIG. 5.

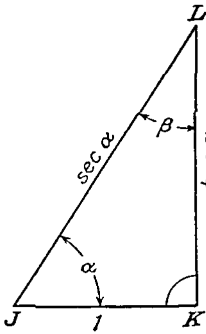


FIG. 6.

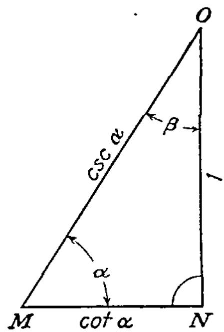


FIG. 7.

RECIPROCAL FUNCTIONS

In a compound-angle problem the function of the angle to be determined is usually stated in the form of a fraction having for its numerator and denominator trigonometric functions of the known angles. When a fraction is stated in terms of the trigonometric functions, the same result can be obtained by multiplying by the reciprocal of the function as when dividing by the function. This is true since the reciprocal of a function is merely the quotient obtained by dividing unity by the function. Since it is easier to multiply by trigonometric functions than to divide by them, the work of compound-angle problems is somewhat simplified if the reciprocals of the trigonometric functions occurring in the denominator are used. It follows from the definitions given of the trigonometric functions and their reciprocal relations that the following pairs of trigonometric functions are reciprocal: sin α and csc α, cos α and sec α, and tan α and cot α.

From the foregoing data it follows that instead of dividing by the sin α we may multiply by the csc α and *vice versa*. In the third edition of Volume I of "Practical Shop Mathematics," the reciprocal relations have been proved and for convenience are restated at this time.

$$\frac{1}{\sin \alpha} = \csc \alpha. \qquad \frac{1}{\csc \alpha} = \sin \alpha.$$

$$\frac{1}{\tan \alpha} = \cot \alpha. \qquad \frac{1}{\cot \alpha} = \tan \alpha.$$

$$\frac{1}{\cos \alpha} = \sec \alpha. \qquad \frac{1}{\sec \alpha} = \cos \alpha.$$

SOLID GEOMETRY

In this text no attempt will be made to prove any of the propositions of solid geometry, but some of the more important propositions, frequently used in solving problems on compound angles, will be stated here. It is suggested that the student become familiar with these statements.

PROPOSITION 47

If two planes intersect each other, their intersection is a straight line.

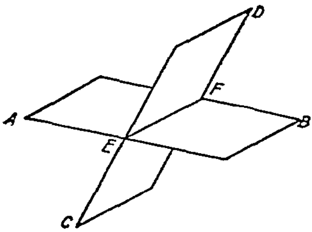


FIG. 8.

In the foregoing figure, the planes AB and CD intersect in the straight line EF.

PROPOSITION 48

If a line is perpendicular to a plane, all lines in the plane passing through the foot of the perpendicular are perpendicular to the line.

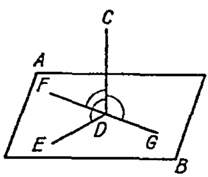


FIG. 9.

If the line CD is perpendicular to the plane AB, the lines ED, FD, and GD are all perpendicular to the line CD.

If EF is perpendicular to the plane AB, the plane CD containing EF is perpendicular to the plane AB.

PROPOSITION 54

If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.

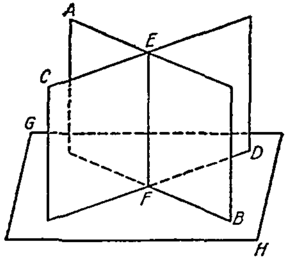


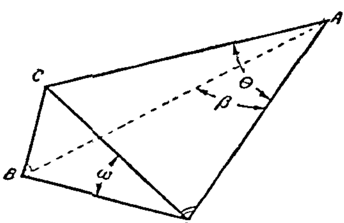
FIG. 15.

If planes AB and CD are both perpendicular to the plane GH, the line of intersection EF is perpendicular to the plane GH.

Definition of a Complete Triangle

When two of the angles in a face of a triangular pyramid can be computed or are given, the faces containing these angles have a definite shape, the form of which is a triangle. This triangle, which has two of its angles given, will be referred to as a complete triangle.

GENERAL PROCEDURE FOR SOLVING TYPE I PROBLEMS



Type I

FIG. 16.

Type I is a triangular pyramid, the four faces of which are all right triangles. When the triangular pyramid is in

the form of Type I, any unknown angle may be computed from any two given angles which lie in adjacent planes. If the two angles lying in each of two adjacent faces of Type I are given, then two of the triangles are complete and the triangular pyramid can be determined.

Example: Let it be required to determine the face angle θ in the triangular pyramid ABC-D. The given parts are the two right angles at B and the two right angles at D (which means that the triangular pyramid is in the form of Type I) and the two face angles ω and β.

Solution: The two complete triangles are ABD and CBD and have a common side BD which may be called unity. With the side BD as unity, the side CD becomes the secant of angle ω, and the side AD becomes the cotangent of the angle β. The sides CD and AD now have definite values and are in the right triangle ACD, which contains the unknown angle θ. By plane trigonometry $\tan\theta = \dfrac{CD}{AD} = \dfrac{\sec\omega}{\cot\beta}$. Instead of dividing by $\cot\beta$, multiply by its reciprocal.

Then $$\tan\theta = \sec\omega\tan\beta.$$

This procedure may be summarized as follows:

1. Letter the vertices of the triangular pyramid.
2. Locate the two complete triangles and let their common edge be equal to unity.
3. Express the two sides of the unknown triangle containing the angle to be computed in terms of the functions of the angles of the two complete triangles.
4. Express a trigonometric function of the angle to be computed in terms of the two sides obtained in step 3.
5. The foregoing function of the angle will be expressed as a fraction. Eliminate the denominator by multiplying the numerator by the reciprocal of the denominator.

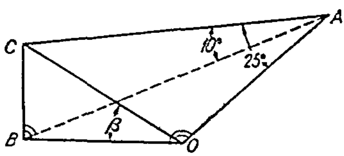


FIG. 17.

If EF is perpendicular to the plane AB, the plane CD containing EF is perpendicular to the plane AB.

PROPOSITION 54

If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane.

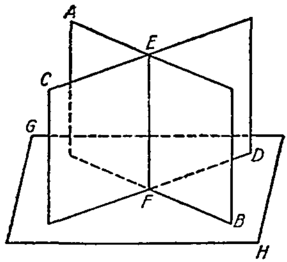


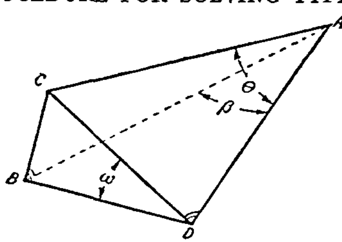
FIG. 15.

If planes AB and CD are both perpendicular to the plane GH, the line of intersection EF is perpendicular to the plane GH.

Definition of a Complete Triangle

When two of the angles in a face of a triangular pyramid can be computed or are given, the faces containing these angles have a definite shape, the form of which is a triangle. This triangle, which has two of its angles given, will be referred to as a complete triangle.

GENERAL PROCEDURE FOR SOLVING TYPE I PROBLEMS

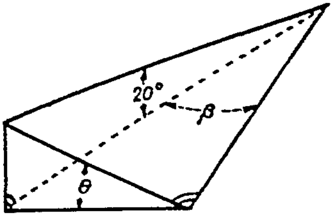


Type I

FIG. 16.

Type I is a triangular pyramid, the four faces of which re all right triangles. When the triangular pyramid is in

PROBLEMS

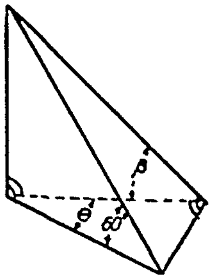


$\theta = 31°$
Ans. $\beta = 37° \, 16' \, 57''$

VARIABLE

1. $\theta = 25°$	2. $\theta = 26°$
3. $\theta = 27°$	4. $\theta = 28°$
5. $\theta = 29°$	6. $\theta = 30°$

1. Determine the angle β.

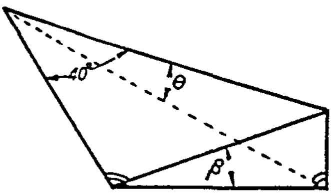


$\theta = 21°$
Ans. $\beta = 61° \, 40' \, 23''$

VARIABLE

1. $\theta = 15°$	2. $\theta = 16°$
3. $\theta = 17°$	4. $\theta = 18°$
5. $\theta = 19°$	6. $\theta = 20°$

2. Determine the angle β.

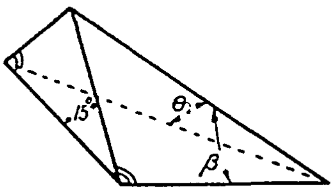


$\theta = 16°$
Ans. $\beta = 25° \, 23' \, 32''$

VARIABLE

1. $\theta = 10°$	2. $\theta = 11°$
3. $\theta = 12°$	4. $\theta = 13°$
5. $\theta = 14°$	6. $\theta = 15°$

3. Determine the angle β.



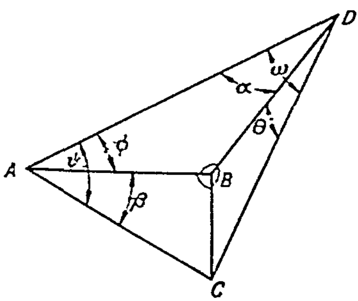
$\theta = 11°$
Ans. $\beta = 47° \, 29' \, 45''$

VARIABLE

1. $\theta = 5°$	2. $\theta = 6°$
3. $\theta = 7°$	4. $\theta = 8°$
5. $\theta = 9°$	6. $\theta = 10°$

4. Determine the angle β.

GENERAL PROCEDURE FOR SOLVING TYPE II PROBLEMS



Type II

FIG. 19.

Type II is a triangular pyramid, three faces of which are right triangles, and the fourth of which, lying in the diagonal plane, is an oblique triangle. Be sure to observe the distinction between the two types I and II. When the triangular pyramid is in the form of Type II, any unknown angle may be computed from any two given angles which lie in adjacent planes except for one special case which will be discussed later.

When two right triangles are complete and perpendicular to a third triangle of the triangular pyramid (Type II) and a face angle in this third triangle is to be computed, the procedure is similar to that given for Type I problems (page 11); but if a face angle in the oblique triangle (fourth face of the triangular pyramid) is to be computed, proceed first by solving for an auxiliary angle in the third right triangle which lies at the same vertex as the required angle. Next split the Type II into two Type I parts, which may be accomplished by constructing a plane perpendicular to one of the sides of the auxiliary angle and passing through the intersection of the two complete right triangles.

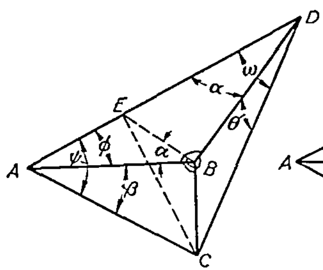


Fig 20.

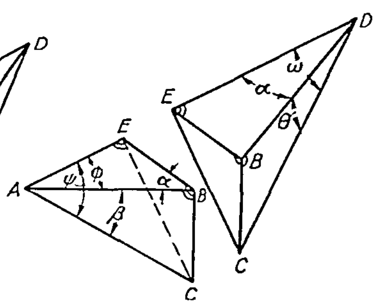


Fig. 21.

Figure 21 shows the pyramid of Fig. 20 divided into two parts by an auxiliary plane, each part being a pyramid of Type I. Note that angle *EBC* is a right angle and that the angles *ABE* and *BDE* are equal (Proposition 48 and Corollary to Proposition 30 of Vol. I).

When two angles at the same vertex of a solid in the form of Type II are given, the third angle at that vertex may be

computed by dividing the solid (now in the form of Type II) into two parts as shown in Fig. 21, each part having the form of Type I, and then computing the required angle by the procedure already given for Type I.

Example: Determine the face angle ω in the triangular pyramid ABC-D. The given parts are the three right angles at B (which means that the triangular pyramid is in the form of Type II) and the two face angles β and θ.

Solution: Since the given face angles β and θ are not both at the same vertex as the required angle ω, it is necessary to first solve for angle α.

Using the foregoing procedure given for Type I:

1. Pyramid ABC-D.
2. BC is common to the two complete triangles ABC and DBC.

Hence let BC = unity.

3. $AB = \cot\beta$ and $BD = \cot\theta$.
4. $\tan\alpha = \dfrac{AB}{BD} = \dfrac{\cot\beta}{\cot\theta}$.
5. $\tan\alpha = \cot\beta\tan\theta$.

In Fig. 14, the pyramid BCE-D is of Type I with face angles α and θ given. Solve for ω by the usual Type I procedure.

Formulas for Type II

For practice the student should verify the following formulas which result from a Type II figure:

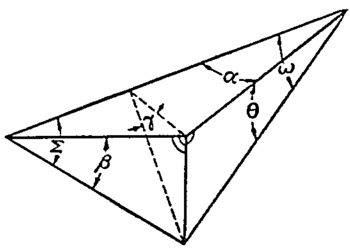


FIG. 22.

Given	To find	Formula
β and θ. .	α	$\tan\alpha = \cot\beta\tan\theta$
γ and θ .	α	$\sin\alpha = \cot\gamma\tan\theta$
ω and θ . .	α	$\cos\alpha = \cos\omega\sec\theta$
ω and γ	α	$\tan\alpha = \cos\gamma\tan\omega$
α and β.	θ	$\tan\theta = \tan\beta\tan\alpha$
α and γ	θ	$\tan\theta = \tan\gamma\sin\alpha$
ω and γ	θ	$\sin\theta = \sin\gamma\sin\omega$
α and ω	θ	$\cos\theta = \sec\alpha\cos\omega$
α and θ	γ	$\cot\gamma = \sin\alpha\cot\theta$
ω and θ	γ	$\sin\gamma = \sin\theta\csc\omega$
α and β	γ	$\cot\gamma = \cos\alpha\cot\beta$
α and ω	γ	$\cos\gamma = \tan\alpha\cot\omega$
α and θ	β	$\tan\beta = \tan\theta\cot\alpha$
α and γ	β	$\tan\beta = \tan\gamma\cos\alpha$
α and γ	ω	$\cot\omega = \cot\alpha\cos\gamma$
α and θ	ω	$\cos\omega = \cos\alpha\cos\theta$
γ and θ	ω	$\sin\omega = \csc\gamma\sin\theta$
β and α	Σ	$\cos\Sigma = \sin\alpha\cos\beta$
β and γ	Σ	$\sin\Sigma = \csc\gamma\sin\beta$
α and γ	Σ	$\tan\Sigma = \sec\gamma\cot\alpha$

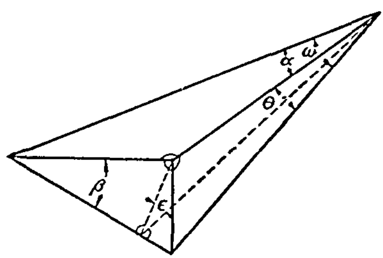


FIG. 23.

Given	To find	Formula
β and θ. .	ϵ	$\cot\epsilon = \cos\beta\tan\theta$
β and α	ϵ	$\cot\epsilon = \sin\beta\tan\alpha$

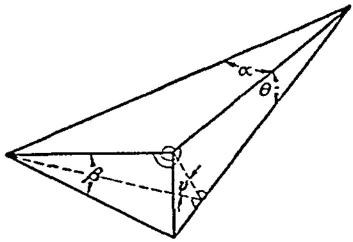
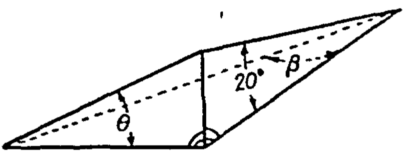


FIG. 24.

Given	To find	Formula
β and θ...........	ψ	$\cot \psi = \cos \theta \tan \beta$
α and θ...........	ψ	$\tan \psi = \tan \alpha \csc \theta$

PROBLEMS

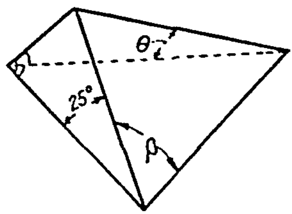


$\theta = 31°$
Ans. $\beta = 31° \; 12' \; 21''$

VARIABLE

1. $\theta = 25°$ 2. $\theta = 26°$
3. $\theta = 27°$ 4. $\theta = 28°$
5. $\theta = 29°$ 6. $\theta = 30°$

1. Determine the angle β.

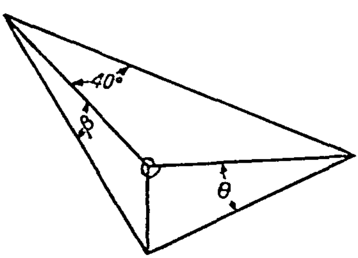


$\theta = 22°$
Ans. $\beta = 53° \; 35' \; 46''$

VARIABLE

1. $\theta = 16°$ 2. $\theta = 17°$
3. $\theta = 18°$ 4. $\theta = 19°$
5. $\theta = 20°$ 6. $\theta = 21°$

2. Determine the angle β.

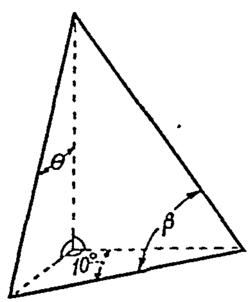


$\theta = 36°$
Ans. $\beta = 31° \; 22' \; 4''$

VARIABLE

1. $\theta = 30°$ 2. $\theta = 31°$
3. $\theta = 32°$ 4. $\theta = 33°$
5. $\theta = 34°$ 6. $\theta = 35°$

3. Determine the angle β.



$\theta = 11°$
Ans. $\beta = 43° \; 9' \; 50''$

VARIABLE

1. $\theta = 5°$ 2. $\theta = 6°$
3. $\theta = 7°$ 4. $\theta = 8°$
5. $\theta = 9°$ 6. $\theta = 10°$

4. Determine the angle β.

Special Case of Type II

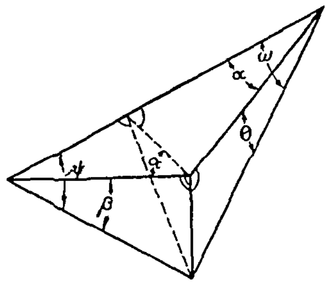


FIG. 25

If one of the given angles lies in the oblique triangle, which is not a complete triangle, and the other given angle lies in the right triangle opposite the vertex of the first given angle, solve for the angle α by the following formulas (which are derived by the use of algebra), and then proceed in the usual manner to solve for any other desired angle.

For β and ω given,

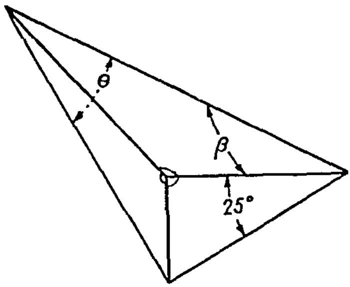
$$\cot^2 \alpha = \frac{\sec^2 \beta + \sqrt{\sec^4 \beta + 4 \tan^2 \omega \tan^2 \beta}}{2 \tan^2 \omega}.$$

For θ and ψ given,

$$\tan^2 \alpha = \frac{\sec^2 \theta + \sqrt{\sec^4 \theta + 4 \tan^2 \psi \tan^2 \theta}}{2 \tan^2 \psi}.$$

After substituting in the values and thus obtaining $\cot^2 \alpha$ by one of the foregoing formulas, extract the square root of the right-hand member which will give the value of $\cot \alpha$, from which the angle α can be obtained.

PROBLEMS



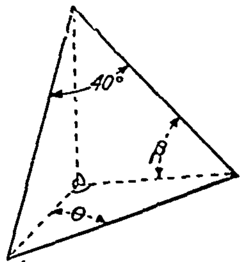
$\theta = 36°$

Ans. $\beta = 57° \; 32' \; 48''$

VARIABLE

1. $\theta = 30°$ 2. $\theta = 31°$
3. $\theta = 32°$ 4. $\theta = 33°$
5. $\theta = 34°$ 6. $\theta = 35°$

1. Determine the angle β.



$\theta = 56°$

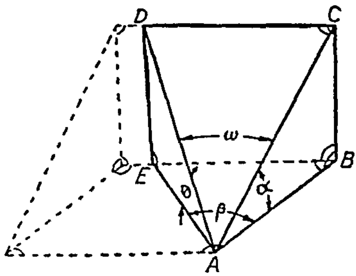
Ans. $\beta = 56° \; 50' \; 5''$

VARIABLE

1. $\theta = 50°$ 2. $\theta = 51°$
3. $\theta = 52°$ 4. $\theta = 53°$
5. $\theta = 54°$ 6. $\theta = 55°$

2. Determine the angle β.

GENERAL PROCEDURE FOR SOLVING TYPE III PROBLEMS



Type III

FIG. 26.

Type III is a quadrangular pyramid having five faces, four of which are right triangles, and the fifth of which is a rectangle. In this type, any unknown angle may be computed from any two given angles.

It can be readily shown that a Type III problem may be reduced to a problem of Type I. However, it is more convenient to solve this type directly by the following method:

Observe that the figure *BCDE* is a rectangle, and hence that the opposite sides are equal.

To compute an angle in a Type III figure, having given two angles which lie in adjacent right triangles, proceed as follows:

In the rectangular pyramid let the common edge of the two complete triangles be unity. State the common edge of the triangle which contains the angle to be computed and the complete triangle adjacent to this triangle, by the function of the given angle. Next state the common edge of the rectangle and the other complete triangle in terms of the function of the given angle of this complete triangle and use this function to represent the opposite side of the rectangle. Express the function of the angle to be computed in terms of the functions which represent the sides of the triangle which contains the angle to be computed.

To compute an angle in a Type III figure, having given two angles lying in right triangles which form the opposite sides of the pyramid, proceed as follows:

In the rectangular pyramid let the equal edges of the two complete triangles be unity. State two edges of the triangle, which contains the angle to be computed, by functions of the given angles of the complete triangles. Express the function of the angle to be computed in terms of the functions which represent the sides of the triangle which contains the angle to be computed.

Example: In the Type III quadrangular pyramid *BCDE-A*, assume that the angles α and β in adjacent right triangles are given. Determine the angle ω.

Solution: The procedure may be carried out in the same order as for Type I.

1. Pyramid *BCDE-A*.

2. Let the common edge AB of the two complete right triangles equal unity.

3. $BE = DC = \tan \beta$, and $AC = \sec \alpha$.

4. $\tan \omega = \dfrac{DC}{AC} = \dfrac{\tan \beta}{\sec \alpha}$.

5. $\tan \omega = \tan \beta \cos \alpha$.

A numerical problem will be worked out to further illustrate is method.

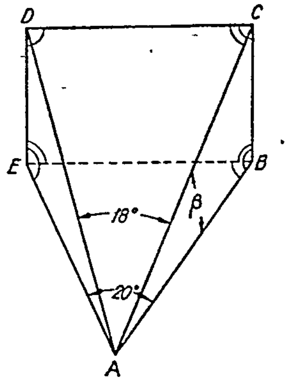


FIG. 27.

Example: Given angle $BAE = 20°$ and the opposite angle $CAD = 18°$.

Determine the angle β.

Solution: 1. Pyramid $BCDE$-A.

2. $BE = DC$; since they are equal and each lies in a complete triangle, they may be called the common edge of the two triangles ABE and ACD $\therefore BE = DC = 1$.

3. $AB = \cot 20°$ and $AC = \cot 18°$.

4. $\cos \beta = \dfrac{AB}{AC} = \dfrac{\cot 20°}{\cot 18°}$.

5. $\cos \beta = \cot 20° \tan 18° = 2.7475 \times .32492 = .89272$.

$\therefore \beta$ is the angle whose cosine is .89272 or $\beta = 26° 47'$.

Formulas for Type III

The student should verify the following two sets of formulas which apply to Type III figures:

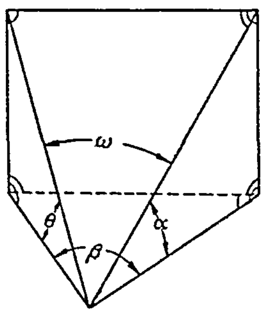


FIG. 28.

Given	To find	Formula
α and β.........	θ	$\tan\theta = \tan\alpha\cos\beta$
α and β.........	ω	$\tan\omega = \tan\beta\cos\alpha$
α and ω.........	θ	$\sin\theta = \sin\alpha\cos\omega$
α and ω.........	β	$\tan\beta = \tan\omega\sec\alpha$
α and θ.........	β	$\cos\beta = \cot\alpha\tan\theta$
α and θ.........	ω	$\cos\omega = \csc\alpha\sin\theta$
β and ω.........	α	$\cos\alpha = \cot\beta\tan\omega$
β and ω.........	θ	$\cos\theta = \csc\beta\sin\omega$
β and θ.........	α	$\cot\alpha = \cos\beta\cot\theta$
β and θ.........	ω	$\sin\omega = \sin\beta\cos\theta$
ω and θ.........	α	$\sin\alpha = \sin\theta\sec\omega$
ω and θ.........	β	$\sin\beta = \sin\omega\sec\theta$

An important application of Type III is in boring an angular hole in a block. The line AB is the boring axis.

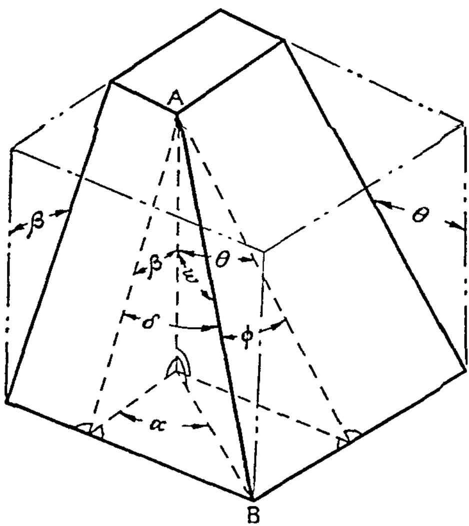
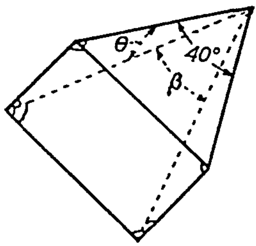


FIG. 29.

Given	To find	Formula
θ and β.........	δ	$\tan\delta = \tan\theta\cos\beta$
θ and β.........	ϕ	$\tan\phi = \tan\beta\cos\theta$
δ and β........	ω	$\cos\omega = \cos\beta\cos\delta$
θ and ϕ........	ω	$\cos\omega = \cos\theta\cos\phi$
α and β.........	ω	$\tan\omega = \sec\alpha\tan\beta$
α and θ.........	ω	$\tan\omega = \csc\alpha\tan\theta$

Note: If θ and β are given and ω is required, δ or α should first be obtained.

PROBLEMS

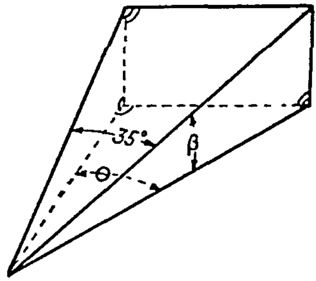


$\theta = 26°$
Ans. $\beta = 43° 2' 0''$

VARIABLE

1. $\theta = 20°$	**2.** $\theta = 21°$
3. $\theta = 22°$	**4.** $\theta = 23°$
5. $\theta = 24°$	**6.** $\theta = 25°$

1. Determine the angle β.

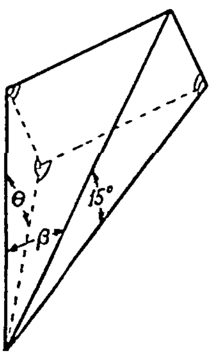


$\theta = 46°$
Ans. $\beta = 37° 7' 6''$

VARIABLE

1. $\theta = 40°$	**2.** $\theta = 41°$
3. $\theta = 42°$	**4.** $\theta = 43°$
5. $\theta = 44°$	**6.** $\theta = 45°$

2. Determine the angle β.

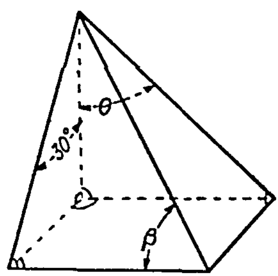


$\theta = 31°$
Ans. $\beta = 59° 50' 0''$

VARIABLE

1. $\theta = 25°$	**2.** $\theta = 26°$
3. $\theta = 27°$	**4.** $\theta = 28°$
5. $\theta = 29°$	**6.** $\theta = 30°$

3. Determine the angle β.



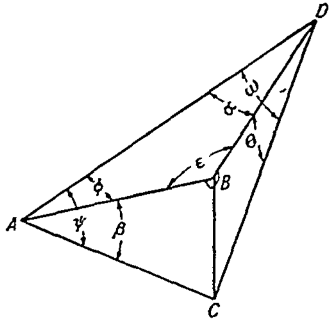
$\theta = 51°$
Ans. $\beta = 43° 4' 39''$

VARIABLE

1. $\theta = 45°$	**2.** $\theta = 46°$
3. $\theta = 47°$	**4.** $\theta = 48°$
5. $\theta = 49°$	**6.** $\theta = 50°$

4. Determine the angle β.

GENERAL PROCEDURE FOR SOLVING TYPE IV PROBLEMS



Type IV
FIG. 30

Type IV is a triangular pyramid, of which two faces are right triangles and two are oblique triangles, and the third angle ϵ at the vertex where the two right angles are located is an obtuse angle. In Type IV, any unknown angle may be computed if any three angles lying in three different faces are given, or if two angles in one of the oblique faces and one angle in a right triangle are given.

Special Case —If the three given angles are at three different vertices and two of them lie in separate oblique triangles, the solution is algebraic in nature and outside the scope of this book.

In order to obtain a solution for some of the cases of Type IV, it is necessary to divide the triangular pyramid into two parts, in the same manner as for Type II, the division being shown below:

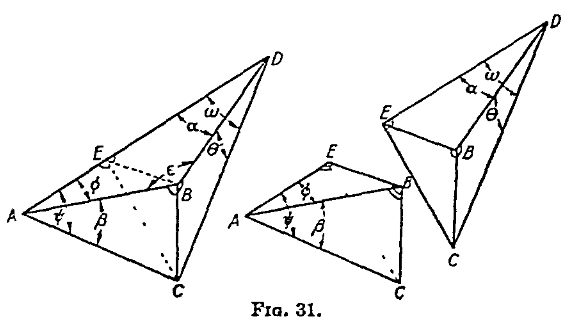


FIG. 31.

When two of the given angles lie in the right triangles and the third given angle (which lies in one of the oblique triangles) is at the same vertex as one of the other given angles, split the triangular pyramid into two parts, each part having the form of Type I as in Fig. 31, and solve for the unknown angle by the usual procedure for Type I.

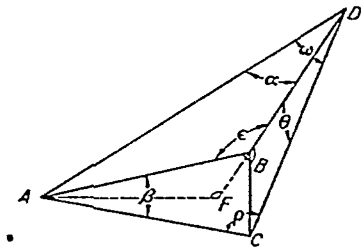


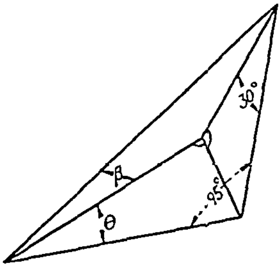
FIG. 32.

When two of the given angles lie in the right triangles and the third given angle is either the obtuse angle ϵ or ρ it is most convenient to solve first for the angle α, or the angle ω, and then for the unknown angles. In order to compute the angle α of the oblique triangle ABD, draw the line AF perpendicular to the line DB extended. The common edge of the two complete triangles is BC. If this edge is considered as unity, then:

$$AB = \cot\beta.$$
$$DB = \cot\theta.$$
$$BF = AB\cos\delta = \cot\beta\cos\delta \text{ where } \delta = 180^\circ - \epsilon.$$
$$DF = DB + BF = \cot\theta + \cot\beta\cos\delta.$$
$$AF = AB\sin\delta = \cot\beta\sin\delta.$$
$$\cot\alpha = \frac{DF}{AF} = \frac{\cot\theta + \cot\beta\cos\delta}{\cot\beta\sin\delta}$$
$$= \frac{\cot\theta}{\cot\beta\sin\delta} + \frac{\cot\beta\cos\delta}{\cot\beta\sin\delta}$$
$$= \cot\theta\tan\beta\csc\delta + \cot\delta.$$

With α thus obtained, any other unknown angle of this figure may be computed by a Type I solution.

PROBLEMS

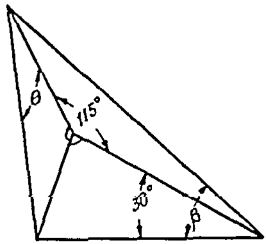


$\theta = 27°$
Ans. $\beta = 30° 40' 4''$

VARIABLE

1. $\theta = 15°$ 2. $\theta = 17°$
3. $\theta = 19°$ 4. $\theta = 21°$
5. $\theta = 23°$ 6. $\theta = 25°$

1. Determine the angle β.

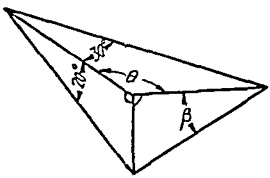


$\theta = 28°$
Ans. $\beta = 44° 6' 57''$

VARIABLE

1. $\theta = 16°$ 2. $\theta = 18°$
3. $\theta = 20°$ 4. $\theta = 22°$
5. $\theta = 24°$ 6. $\theta = 26°$

2. Determine the angle β.



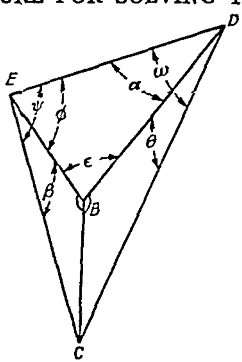
VARIABLE

No.	Sym.	Value
1	θ	111°
2	θ	113°
3	θ	115°
4	θ	117°
5	θ	119°
6	θ	121°

$\theta = 123°$
Ans. $\beta = 18° 17' 15''$

3. Determine the angle β.

GENERAL PROCEDURE FOR SOLVING TYPE V PROBLEMS



Type V
FIG. 33.

Type V is a triangular pyramid, of which two faces are right triangles, and two are oblique triangles, and the third angle ϵ at the vertex where the two right angles are located is an acute angle. Note that Type V is the same as Type IV, except that the angle ϵ is acute instead of obtuse.

Special Case.—When both ϵ and ϕ are acute angles, the solution is the same as Type IV.

In Type V, any unknown angle may be computed if any three angles lying in three different faces, or if two angles in one of the oblique faces and one angle in a right triangle, are given.

The same special case exists in Type V as was mentioned in Type IV.

In order to obtain a solution for some of the cases of Type V, it is necessary to add a Type I portion as shown in the figure below, which will make the whole figure in the form of Type I.

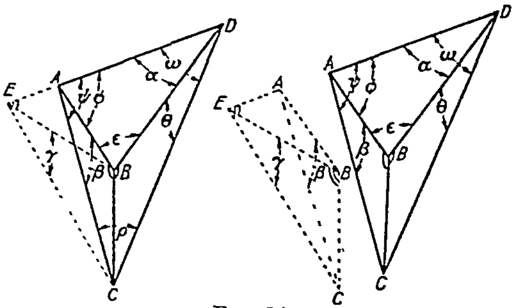


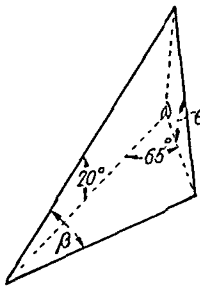
Fig. 34.

When two of the given angles lie in the right triangles and the third given angle (which lies in one of the oblique triangles) is at the same vertex as one of the other given angles, add the Type I pyramid *EBA-C* to form the large Type I figure *DEB-C*. Solve for the unknown angle by the usual procedure for Type I. In some cases, it will be necessary to use the supplements of ψ or ϕ in order to solve for other angles.

When two of the given angles lie in the right triangles and the third given angle is either the acute angle ϵ or ρ, it is best, as in Type IV, first to solve for the angle α or the angle ω, and then for the unknown angle.

The formula for computing the angle α is derived by a method similar to that used in Type IV and may be shown by the student to be

PROBLEMS

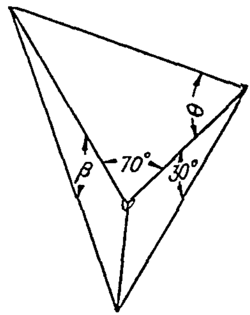


$\theta = 48°$
Ans. $\beta = 27° \; 19' \; 50''$

Variable

1. $\theta = 42°$	**2.** $\theta = 43°$
3. $\theta = 44°$	**4.** $\theta = 45°$
5. $\theta = 46°$	**6.** $\theta = 47°$

1. Determine the angle β.



$\theta = 65°$
Ans. $\beta = 24° \; 14' \; 58''$

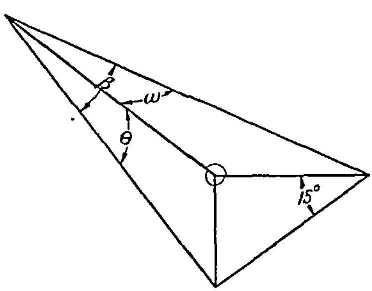
Variable

1. $\theta = 59°$	**2.** $\theta = 60°$
3. $\theta = 61°$	**4.** $\theta = 62°$
5. $\theta = 63°$	**6.** $\theta = 64°$

2. Determine the angle β.

The following problems are designed to give the student further practice in solving problems of the various types. The type figure involved should first be recognized, and then the solution carried out by a procedure similar to that used for Type I problems.

PROBLEMS



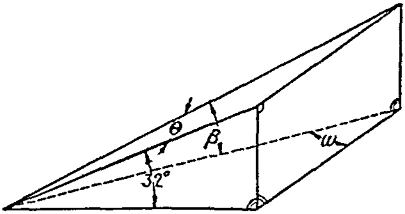
Variable

No.	Sym.	Value
1	θ	12°
2	θ	13°
3	θ	14°
4	θ	15°
5	θ	16°
6	θ	17°

$\theta = 11°$

$$Ans. \begin{cases} \beta = 37° \; 23' \; 16'' \\ \omega = 35° \; 57' \; 28'' \end{cases}$$

1. Determine the angle β. **2.** Determine the angle ω.



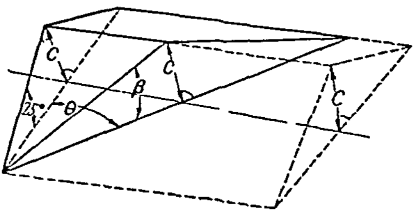
Variable		
No.	Sym.	Value
1	θ	34°
2	θ	36°
3	θ	38°
4	θ	40°
5	θ	42°
6	θ	44°

$\theta = 32°$

Ans. $\begin{cases} \beta = 26° \; 42' \; 16'' \\ \omega = 53° \; 36' \; 52'' \end{cases}$

3. Determine the angle β.

4. Determine the angle ω.

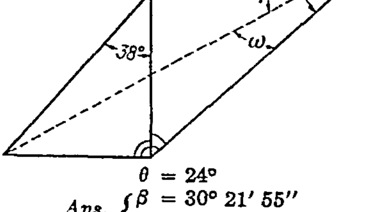


Variable		
No.	Sym.	Value
1	θ	38°
2	θ	40°
3	θ	42°
4	θ	44°
5	θ	46°
6	θ	48°

$\theta = 36°$

Ans. $\beta = 20° \; 40' \; 9''$

5. Determine the angle β.



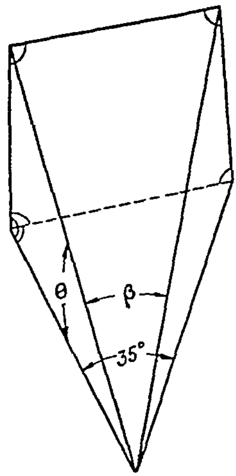
Variable		
No.	Sym.	Value
1	θ	26°
2	θ	28°
3	θ	30°
4	θ	32°
5	θ	34°
6	θ	36°

$\theta = 24°$

Ans. $\begin{cases} \beta = 30° \; 21' \; 55'' \\ \omega = 19° \; 10' \; 47'' \end{cases}$

6. Determine the angle β.

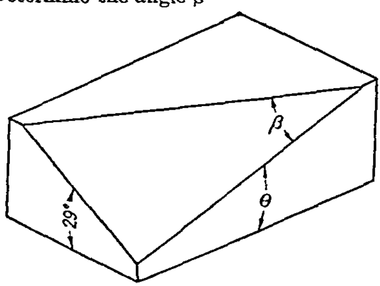
7. Determine the angle ω.



Variable		
No.	Sym.	Value
1	θ	13°
2	θ	15°
3	θ	17°
4	θ	19°
5	θ	21°
6	θ	23°

$\theta = 11°$
Ans. $\beta = 34° 30' 8''$

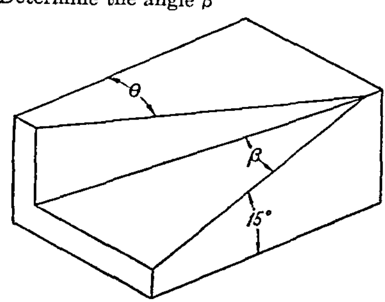
8. Determine the angle β



Variable		
No.	Sym.	Value
1	θ	22°
2	θ	23°
3	θ	24°
4	θ	25°
5	θ	26°
6	θ	27°

$\theta = 21°$
Ans. $\beta = 39° 52' 6''$

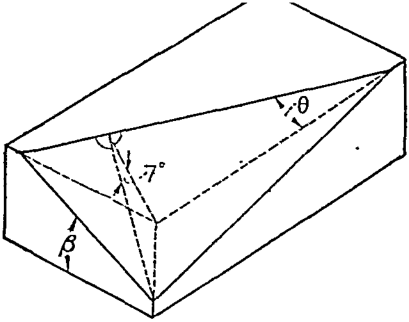
9. Determine the angle β



Variable		
No.	Sym.	Value
1	θ	27°
2	θ	29°
3	θ	31°
4	θ	33°
5	θ	35°
6	θ	37°

$\theta = 25°$
Ans. $\beta = 24° 14' 49''$

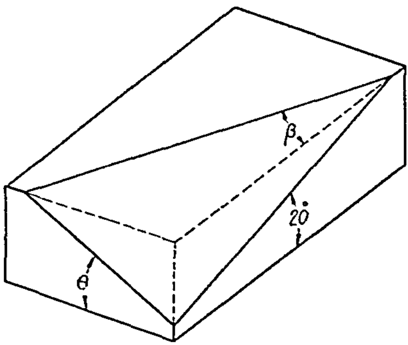
10. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	32°
2	θ	33°
3	θ	34°
4	θ	35°
5	θ	36°
6	θ	37°

$\theta = 31°$
Ans. $\beta = 6° \, 0' \, 10''$

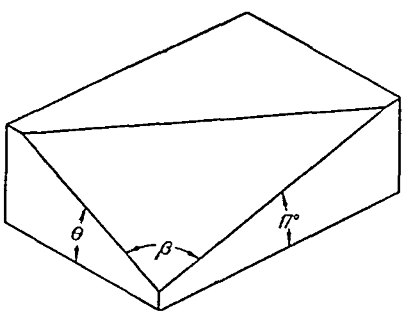
11. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	34°
2	θ	36°
3	θ	38°
4	θ	41°
5	θ	43°
6	θ	45°

$\theta = 32°$
Ans. $\beta = 30° \, 13' \, 9''$

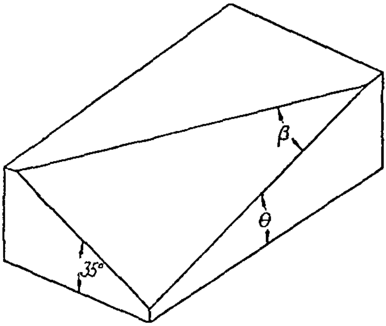
12. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	12°
2	θ	14°
3	θ	16°
4	θ	18°
5	θ	20°
6	θ	22°

$\theta = 24°$
Ans. $\beta = 83° \, 10' \, 14''$

13. Determine the angle β.

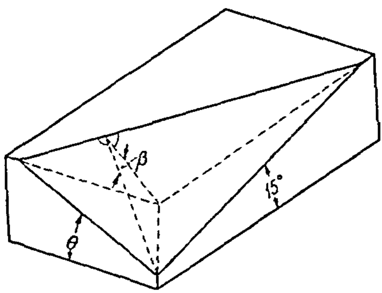


VARIABLE		
No.	Sym.	Value
1	θ	29°
2	θ	31°
3	θ	34°
4	θ	37°
5	θ	40°
6	θ	43°

$\theta = 23°$

Ans. $\beta = 38° \, 4' \, 33''$

14. Determine the angle β.

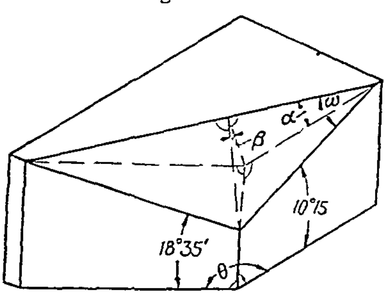


VARIABLE		
No.	Sym.	Value
1	θ	9°
2	θ	10°
3	θ	11°
4	θ	12°
5	θ	13°
6	θ	14°

$\theta = 8°$

Ans. $\beta = 16° \, 50' \, 5''$

15. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	122°
2	θ	133°
3	θ	144°
4	θ	155°
5	θ	166°
6	θ	177°

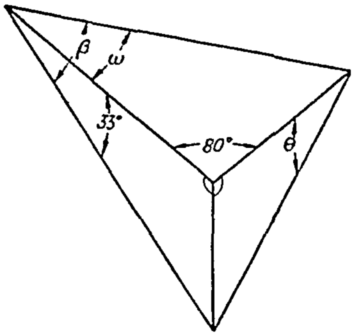
$\theta = 111°$

Ans. $\begin{cases} \alpha = 22° \, 49' \, 46'' \\ \beta = 24° \, 59' \, 20'' \\ \omega = 24° \, 54' \, 40'' \end{cases}$

16. Determine the angle α.

17. Determine the angle β.

18. Determine the angle ω.

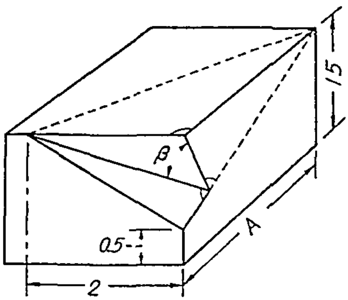


Variable		
No.	Sym.	Value
1	θ	12°
2	θ	14°
3	θ	16°
4	θ	18°
5	θ	20°
6	θ	22°

$\theta = 10°$

Ans. $\begin{cases} \beta = 85° \, 14' \, 34'' \\ \omega = 84° \, 19' \, 30'' \end{cases}$

19. Determine the angle β. **20.** Determine the angle ω.

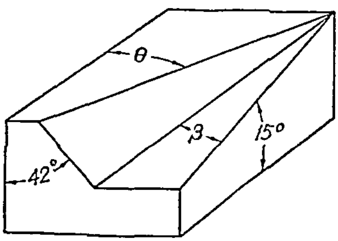


Variable		
No.	Sym.	Value
1	A	4.750
2	A	4.875
3	A	5.000
4	A	5.125
5	A	5.250
6	A	5.375

$A = 5.500$

Ans. $\beta = 63° \, 48' \, 20''$

21. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	32°
2	θ	33°
3	θ	34°
4	θ	35°
5	θ	36°
6	θ	37°

$\theta = 38°$

Ans. $\beta = 27° \, 32' \, 49''$

22. Determine the angle β.

Special Cases Type IV and V

The known angles are ϕ, α, and ρ with angles ϕ and α each less than 90°.

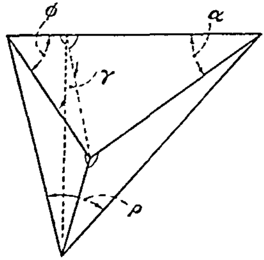


FIG. 35.

To solve for any other angle in Fig. 35, first compute the angle γ in the auxiliary plane by the following formula, and proceed with Type I in the usual way for computing any desired angle.

$$\cos \gamma = \tfrac{1}{2}(\sqrt{A^2 + B} - A).$$

Where $A = \cot \rho(\tan \alpha + \tan \phi)$, and $B = 4 \tan \alpha \tan \phi$.

The known angles in Fig. 36 are ϕ, α, and either θ or β with the angles ϕ and β each less than 90°.

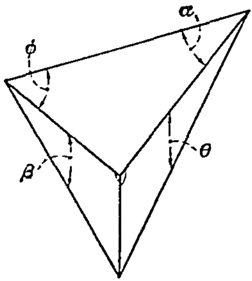
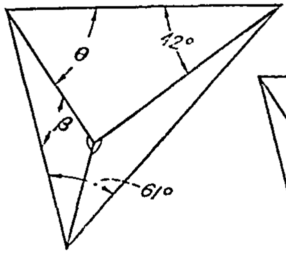


FIG. 36.

To solve for β or θ (whichever angle is not given), use the following formula:

$$\tan \beta = \sin \phi \csc \alpha \tan \theta.$$
$$\tan \theta = \sin \alpha \csc \phi \tan \beta.$$

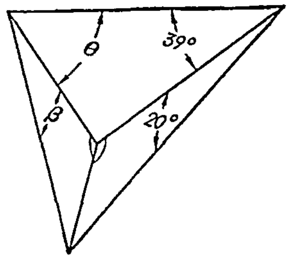


$\theta = 65°$
Ans. $\beta = 35° 51' 31''$

VARIABLE

1. $\theta = 59°$ 2. $\theta = 60°$
3. $\theta = 61°$ 4. $\theta = 62°$
5. $\theta = 63°$ 6. $\theta = 64°$

23. Determine the angle β.



$\theta = 65°$
Ans. $\beta = 27° 39' 42''$

VARIABLE

1. $\theta = 59°$ 2. $\theta = 60°$
3. $\theta = 61°$ 4. $\theta = 62°$
5. $\theta = 63°$ 6. $\theta = 64°$

24. Determine the angle β.

ORTHOGRAPHIC PROJECTION PROBLEMS CONVERTED TO PICTORIAL

The majority of the problems in this text on compound angles are drawn for simplicity in a **pictorial form.** This is not the usual form in which the mechanic is confronted with the solid geometrical shop problems, since he must work from blueprints that have been drawn in the regular mechanical drawing form. Therefore, the authors deem it necessary to devote a few pages to the theory of compound angles and problems drawn by the usual mechanical drawing methods.

A knowledge of orthographic projections is necessary in order to cope thoroughly with the designs drawn on blueprint with the regular mechanical drawing methods. In the study of the solid geometrical shop problems, given in this text, it is assumed that the student has had a knowledge of mechanical drawing. It is the aim of the authors, in the event that the student has forgotten some of the essential theory in mechanical drawing projections, to refresh his mind by giving him a brief review of orthographic projections. This will give him a better conception of the compound-angle problem and will aid him to project with greater confidence the essential points,

which were drawn in a mechanical drawing form, back into a pictorial form.

Most of the problems in this text on compound angles are presented without linear dimensions but are given in terms of their angular magnitude instead, which is sufficient to indicate shape but not size. These problems may be drawn any number of times larger than the original size given in this book, but two or three times larger is sufficient to show all the essential parts of the problem.

ORTHOGRAPHIC PROJECTIONS

To project a point on a plane of reference, according to the mechanical drawing methods, is to have the point move to the plane of reference in such a manner as to travel the least possible distance. Hence, the path of the point will be on a line perpendicular to the plane of reference. Consequently, the point of projection is at the foot of the perpendicular line drawn from the original point to the plane of reference. When several points are projected into three or more planes of reference, and if these planes of reference together with their points of projections are rotated 90° into a single plane (the vertical plane), this method is called orthographic projections. Generally, the essential points of the object are projected into three conventional planes of reference, which are the horizontal, vertical, and profile planes.

The horizontal, vertical, and profile planes, according to the mechanics interpretation, are generally known as the **top, front,** and usually the **right end views,** respectively. When the essential lines of an object are projected into the three views mentioned above, they represent a mechanical drawing made according to the principles of orthographic projections. Naturally, these three views all lie in different planes, but to show them on a flat piece of paper it becomes necessary to rotate the horizontal and profile views into a single plane, which, according to the rules of mechanical drawing, is known as the vertical plane. This is equivalent to rotating the object so that the horizontal and profile views of the object may be seen in the vertical plane. According to the system of ortho-

graphic projections, all planes just prior to the time of rotating them into the vertical plane must be perpendicular to the vertical plane, understanding, of course, that the object has been rotated 90° each time a view is shown in the vertical plane.

When other views are given in order to bring out special detailing, they are called **auxiliary** or **cross-sectional views.** An auxiliary view is a true view of an object which lies in an angular position and which cannot be shown in its true view in any one of the three conventional planes. A true view is an undistorted view where all lines according to the scale of the drawing are represented by their real lengths. Auxiliary and cross-sectional views are also rotated 90° away from the part and into the vertical plane. A cross-sectional view may lie in any plane but usually lies in some angular plane, which in turn is perpendicular to one of the three conventional planes. It is considered to have been cut into two parts with the part in the path of the cross-sectional view removed. Cross-sectional views are always indicated by crosshatching the surface that has been cut. For brevity the term **section** is generally used instead of the term **cross-sectional view.**

In the pursuit of a solution of a compound-angle problem, it is not necessary to draw the entire object as is given on the blueprint into a pictorial drawing form. It is only necessary to draw the essential lines that constitute the compound-angle problem. The essential lines are usually the intersectional lines of the diagonal surfaces that have been projected into two or three of the conventional planes. In the forthcoming compound-angle problems they represent the five basic types, as shown in the beginning of this text, and all of the given lines are essential in order to draw a pictorial representation. In many cases a pictorial conception of the object simplifies the formation of the solution.

The following is an illustrative problem showing how to convert a sketch that has been drawn according to the mechanical drawing rules into a pictorial form. Let it be desired to convert the mechanical drawing shown in Fig. 37 into a pictorial form.

Note: Some of the vertices have been labeled with double letters, meaning that there are two points on the same line. The point in the immediate triangle is labeled by the first letter. The remote point is labeled by the second letter.

The student should sketch very lightly a copy, drawn to twice the size, of the perspective view together with extended lines of the parallelepiped $A'A$, $B'B$, and $C'C$ of Fig. 38.

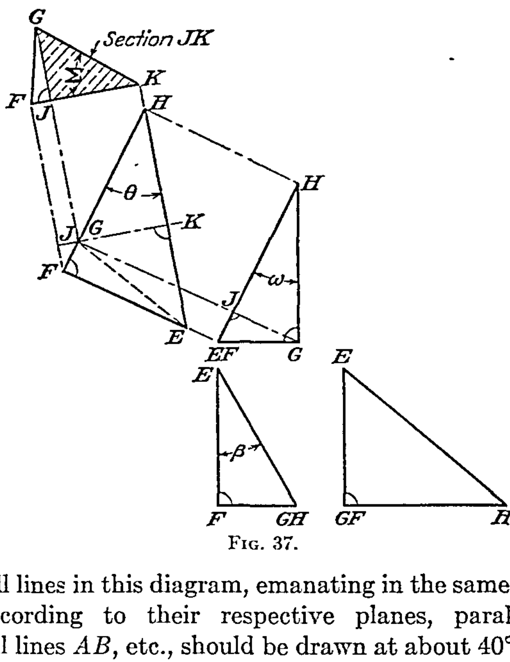


FIG. 37.

Draw all lines in this diagram, emanating in the same direction and according to their respective planes, parallel. The diagonal lines AB, etc., should be drawn at about 40° with the extended line $C'C$. When the above lines are completed, draw an exploded or an extended view of the three pictorial planes, which according to Fig. 38 are the pictorial views of the horizontal, vertical, and profile planes. When the extended planes of the parallelepiped are properly drawn, erase the former part of the sketch consisting of the original parallelepiped and the extended lines leaving only the three xtended views of the three pictorial planes—horizontal, ertical, and profile. The above proportion and entire system

may be used to draw the extended views of the three pictorial planes for all the following problems in this text drawn in a mechanical drawing form.

Next draw to twice the size given in Fig. 37 the three principal views of the object in the horizontal, vertical, and profile planes into their respective extended pictorial planes and also in their rightful location according to the dimensions

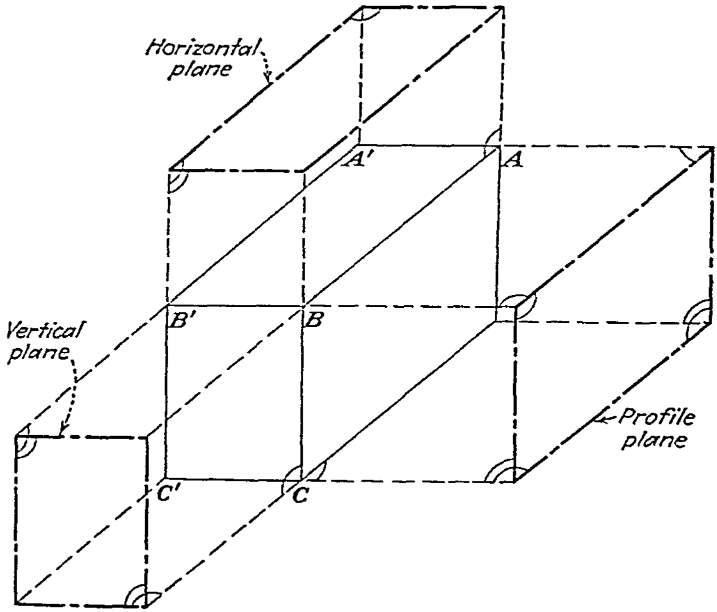


Fig. 38.

labeled R, S, and T, as shown in Fig. 39. Project the points of the triangles now in the extended planes toward the location of the original parallelepiped. The intersections of the dotted lines, which indicate the paths of the projected corresponding points, constitute the principal points of the pictorial object. Connect the points by drawing lines through them according to the corresponding lines in the extended pictorial planes. The result is a pictorial view of the object that previously was drawn by mechanical drawing methods.

In order properly to label the right angles in the pictorial drawing, one must constantly bear in mind that the three

extended pictorial views are always at right angles with each other. Since the line *GH* is shown, in both the horizontal and profile planes, to be perpendicular to the vertical plane, and since the lines *FG* and *EG* both lie in the vertical plane, then by Proposition 48 the line *GH* is perpendicular to both the lines *FG* and *EG*. Likewise, the line *EF* is perpendicular to the two lines *FH* and *FG*. The pictorial drawing shows a

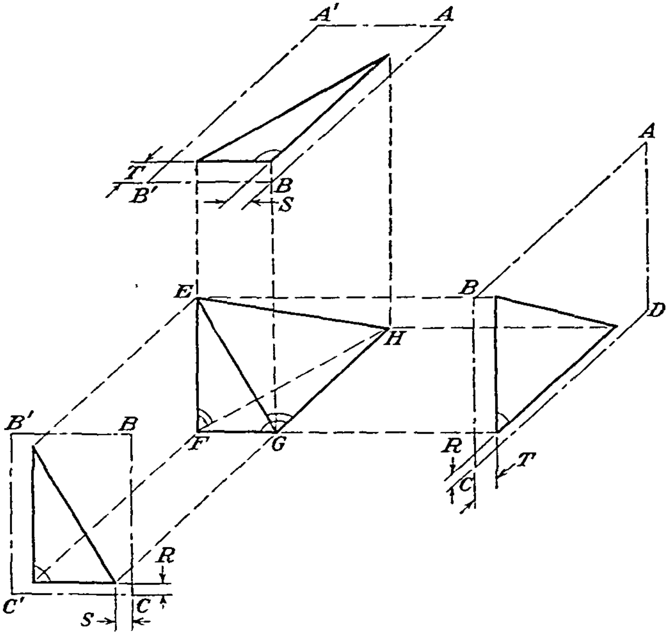


FIG. 39.

triangular pyramid made up of four right triangles, consequently it is a Type I problem.

Figure 40 is a duplication, drawn to twice the original size of the triangular pyramid *GEF-H* shown in Fig. 39, with the cross-sectional plane *GJK* inserted in its proper place. The plane *HFE* in Fig. 37 is an auxiliary view, and since the horizontal view *FGH* of the object has been rotated into the vertical plane, the plane *HFE* has become perpendicular to the vertical plane. It has been rotated 90° into the vertical plane

on a line parallel to *FH* and at a reasonable distance away from the object. The plane *KJG* in Fig. 37 is a cross-sectional view and is cut perpendicular to the plane *HFE* and also to the line *HE*. Therefore, by Proposition 48 the lines *GK* and *JK* in Fig. 40 are both perpendicular to the line *HE*.

Since the line *EF* in Fig. 37 is shown in both the vertical and profile planes to be perpendicular to the horizontal plane, and since the plane *HFE* passes through this line, then by Proposition 53 the plane *HFE* is perpendicular to the horizontal plane *FGH*. Now, since the line *GJ* in Fig. 37 is shown by construc-

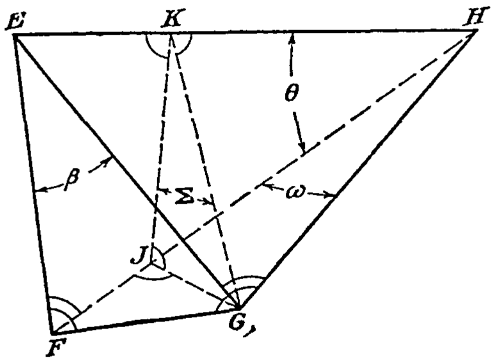


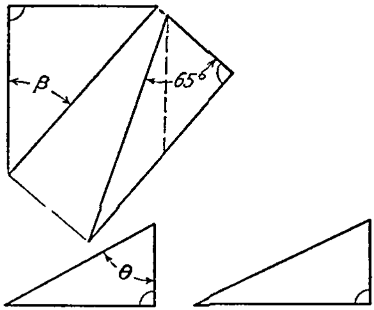
FIG. 40.

tion to be perpendicular to the plane *HFE*, and since the line *JK* lies in the plane *HFE*, then by Proposition 52 the line *JK* is perpendicular to the line *GJ*. Again the triangular pyramid *GJK-H* in Fig 40 is a Type I problem.

All planes with the exception of the plane *EFG* as shown in Fig. 37 have been rotated by a series of 90° into the vertical plane. By this same system the essential lines of all compound-angle problems that have been drawn in a mechanical drawing form may be converted into a pictorial form so that the student may inspect more closely the exact nature of the problem.

The following 55 compound-angle problems have been drawn in a mechanical drawing form and represent the five basic types. They are to be drawn in a pictorial drawing form and then computed for the angles as indicated in the original drawing.

PROBLEMS

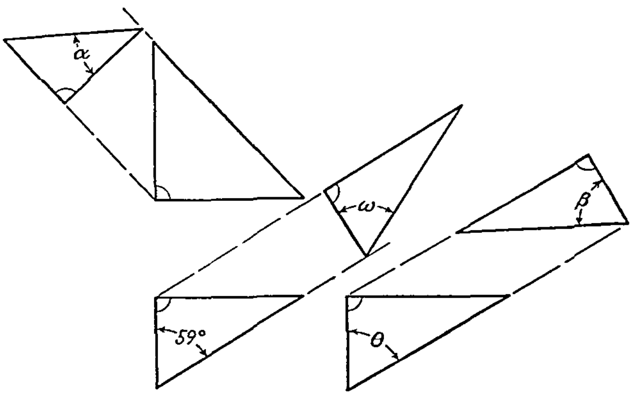


VARIABLE

No.	Sym.	Value
1	θ	57°
2	θ	58°
3	θ	59°
4	θ	60°
5	θ	61°
6	θ	62°

$\theta = 63°$
Ans. $\beta = 66° \ 13' \ 52''$

1. Determine the angle β.



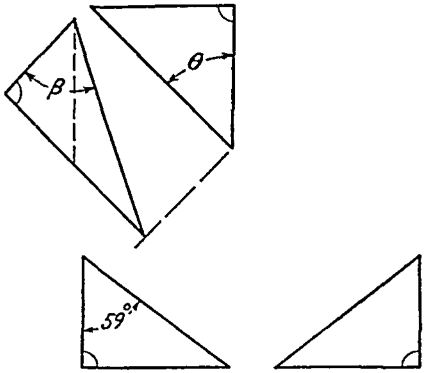
$\theta = 71°$

Ans. $\begin{cases} \alpha = 34° \ 42' \ 13'' \\ \beta = 60° \ 23' \ 52'' \\ \omega = 73° \ 33' \ 22'' \end{cases}$

VARIABLE

1. $\theta = 65°$	**2.** $\theta = 66°$	**3.** $\theta = 67°$
4. $\theta = 68°$	**5.** $\theta = 69°$	**6.** $\theta = 70°$

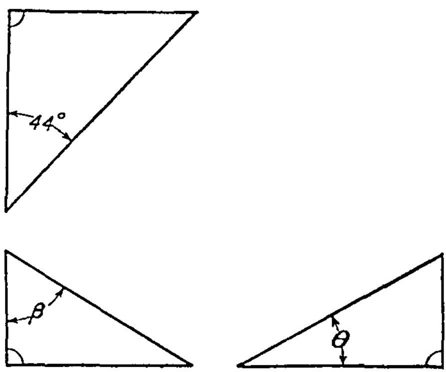
2. Determine the angle α.
3. Determine the angle β.
[illegible]. Determine the angle ω.



Variable		
No.	Sym.	Value
1	θ	35°
2	θ	36°
3	θ	37°
4	θ	38°
5	θ	39°
6	θ	40°

$\theta = 41°$
Ans. $\beta = 68° \, 29' \, 8''$

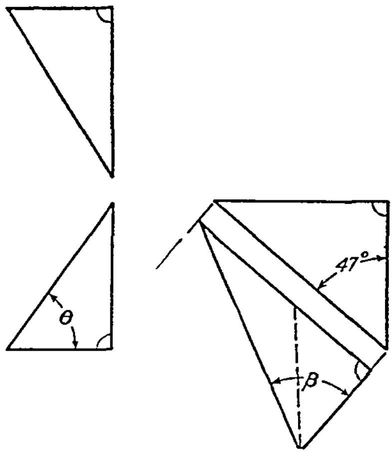
5. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	24°
2	θ	25°
3	θ	26°
4	θ	27°
5	θ	28°
6	θ	29°

$\theta = 30°$
Ans. $\beta = 59° \, 7' \, 35''$

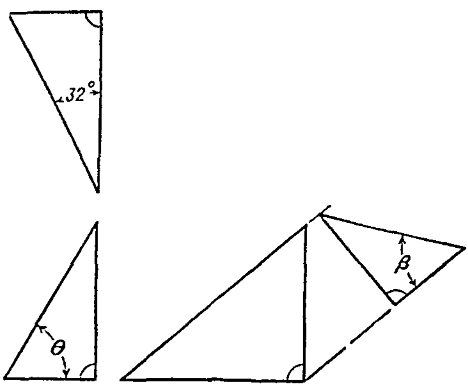
6. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	59°
2	θ	60°
3	θ	61°
4	θ	62°
5	θ	63°
6	θ	64°

$\theta = 65°$
Ans. $\beta = 72° \; 21' \; 29''$

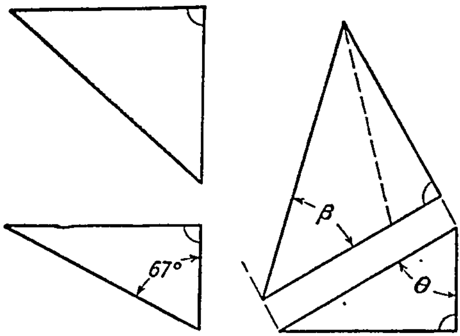
7. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	65°
2	θ	66°
3	θ	67°
4	θ	68°
5	θ	69°
6	θ	70°

$\theta = 71°$
Ans. $\beta = 54° \; 30' \; 38''$

9. Determine the

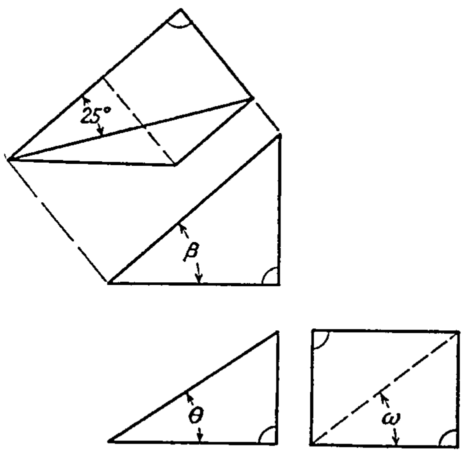


VARIABLE		
No.	Sym.	Value
1	θ	57°
2	θ	58°
3	θ	59°
4	θ	60°
5	θ	61°
6	θ	62°

$\theta = 63°$

Ans. $\beta = 46° \, 55' \, 27''$

9. Determine the angle β.

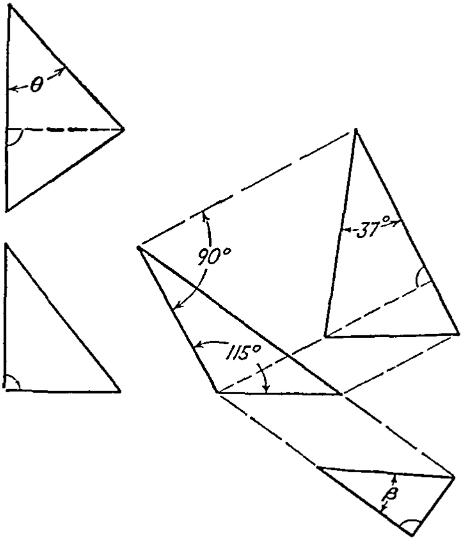


VARIABLE		
No.	Sym.	Value
1	θ	27°
2	θ	28°
3	θ	29°
4	θ	30°
5	θ	31°
6	θ	32°

$\theta = 33°$

Ans. $\begin{cases} \beta = 44° \, 6' \, 23'' \\ \omega = 33° \, 49' \, 19'' \end{cases}$

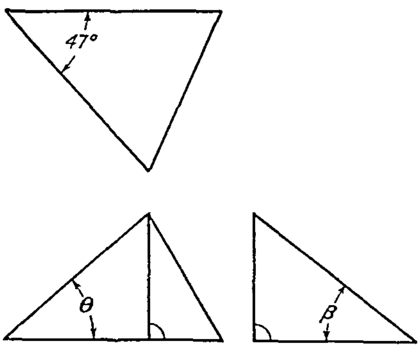
10. Determine the angle β.
11. Determine the angle ω.



VARIABLE		
No.	Sym.	Value
1	θ	40°
2	θ	41°
3	θ	42°
4	θ	43°
5	θ	44°
6	θ	45°

$\theta = 46°$
Ans. $\beta = 30° \ 51' \ 50''$

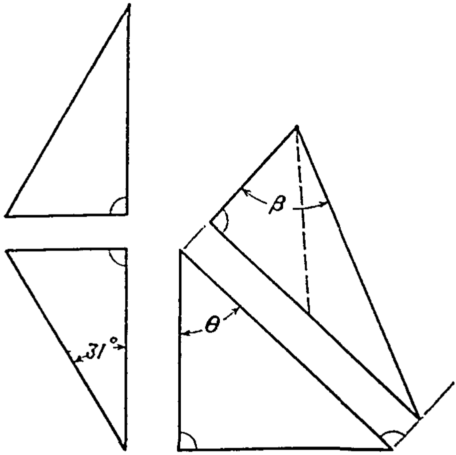
12. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	38°
2	θ	39°
3	θ	40°
4	θ	41°
5	θ	42°
6	θ	43°

$\theta = 44°$
Ans. $\beta = 42° \ 0' \ 13''$

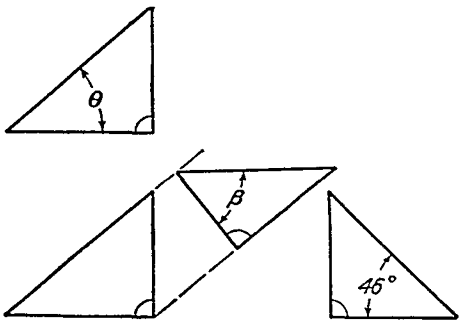
13. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	46°
2	θ	47°
3	θ	48°
4	θ	49°
5	θ	50°
6	θ	51°

$\theta = 52°$
Ans. $\beta = 69° \, 41' \, 57''$

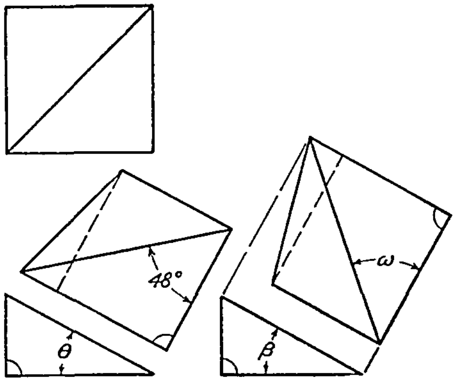
14. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	35°
2	θ	36°
3	θ	37°
4	θ	38°
5	θ	39°
6	θ	40°

$\theta = 41°$
Ans. $\beta = 52° \, 26' \, 42''$

15. Determine the angle β.

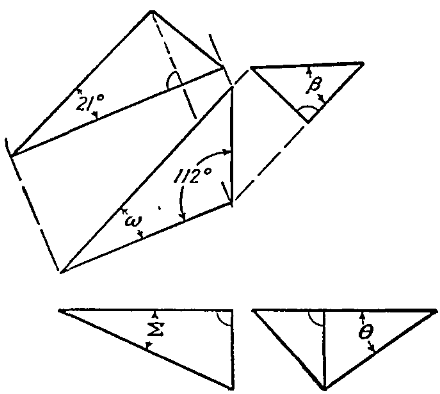


Variable		
No.	Sym.	Value
1	θ	27°
2	θ	28°
3	θ	29°
4	θ	30°
5	θ	31°
6	θ	32°

$\theta = 33°$

Ans. $\begin{cases} \beta = 31° \ 10' \ 8'' \\ \omega = 51° \ 26' \ 46'' \end{cases}$

16. Determine the angle β.
17. Determine the angle ω.

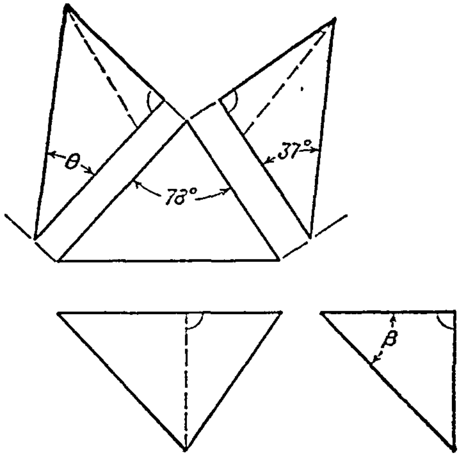


Variable		
No.	Sym.	Value
1	θ	32°
2	θ	33°
3	θ	34°
4	θ	35°
5	θ	36°
6	θ	37°

$\theta = 38°$

Ans. $\begin{cases} \omega = 21° \ 2' \ 34'' \\ \beta = 43° \ 5' \ 17'' \\ \Sigma = 22° \ 29' \ 24'' \end{cases}$

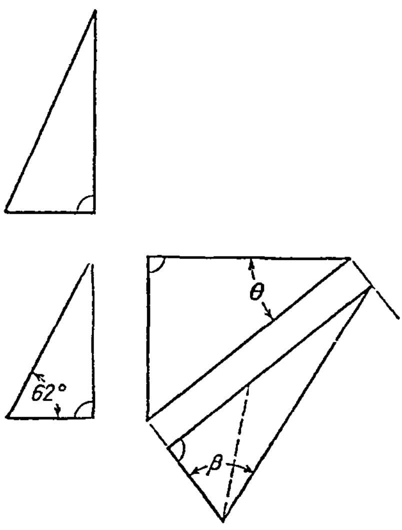
18. Determine the angle ω.
19. Determine the angle β.
20. Determine the angle Σ.



Variable		
No.	Sym.	Value
1	θ	29°
2	θ	30°
3	θ	31°
4	θ	32°
5	θ	33°
6	θ	34°

$\theta = 35°$
Ans. $\beta = 43° \, 6' \, 55''$

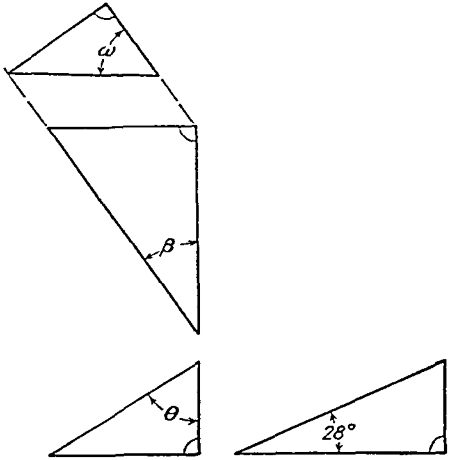
21. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	34°
2	θ	35°
3	θ	36°
4	θ	37°
5	θ	38°
6	θ	39°

$\theta = 40°$
Ans. $\beta = 70° \, 7' \, 50''$

22. Determine the angle β.



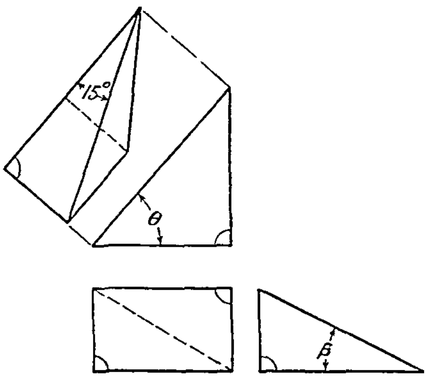
Variable		
No.	Sym.	Value
1	θ	58°
2	θ	59°
3	θ	60°
4	θ	61°
5	θ	62°
6	θ	63°

$\theta = 64°$

Ans. $\begin{cases} \beta = 47° \, 28' \, 12'' \\ \omega = 54° \, 11' \, 19'' \end{cases}$

23. Determine the angle β.

24. Determine the angle ω.

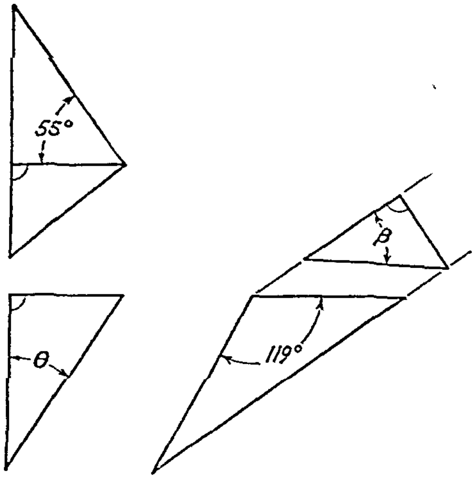


Variable		
No.	Sym.	Value
1	θ	47°
2	θ	48°
3	θ	49°
4	θ	50°
5	θ	51°
6	θ	52°

$\theta = 53°$

Ans. $\beta = 18° \, 32' \, 49''$

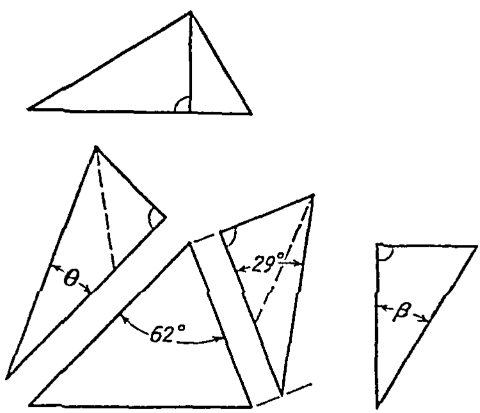
25. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	35°
2	θ	36°
3	θ	37°
4	θ	38°
5	θ	39°
6	θ	40°

$\theta = 41°$
Ans. $\beta = 34° \; 47' \; 29''$

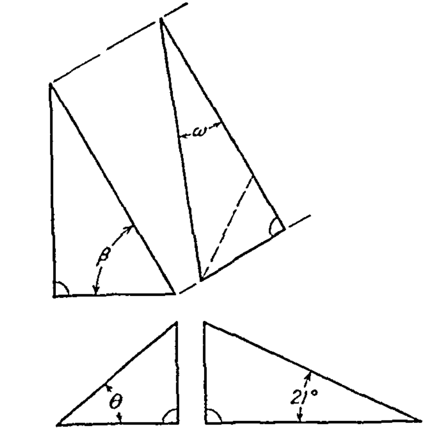
26. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	18°
2	θ	19°
3	θ	20°
4	θ	21°
5	θ	22°
6	θ	23°

$\theta = 24°$
Ans. $\beta = 30° \; 39' \; 2''$

27. Determine the angle β.



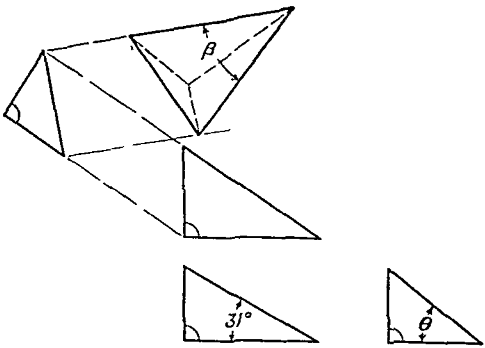
Variable

No.	Sym.	Value
1	θ	35°
2	θ	36°
3	θ	37°
4	θ	38°
5	θ	39°
6	θ	40°

$$\theta = 41^\circ$$

$$\textit{Ans.}\ \begin{cases} \beta = 66^\circ\ 10'\ 28'' \\ \omega = 19^\circ\ 20'\ 55'' \end{cases}$$

28. Determine the angle β.
29. Determine the angle ω.



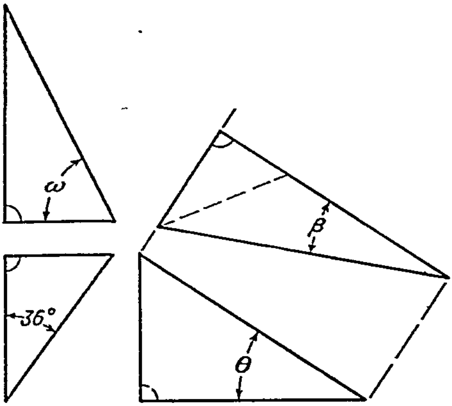
Variable

No.	Sym.	Value
1	θ	37°
2	θ	38°
3	θ	39°
4	θ	40°
5	θ	41°
6	θ	42°

$$\theta = 43^\circ$$

$$\textit{Ans.}\ \beta = 43^\circ\ 54'\ 2''$$

30. Determine the angle β.

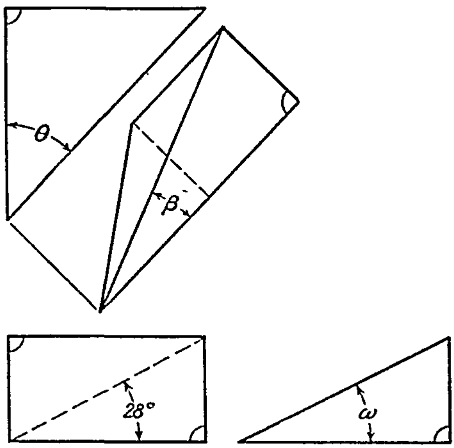


VARIABLE		
No.	Sym.	Value
1	θ	38°
2	θ	39°
3	θ	40°
4	θ	41°
5	θ	42°
6	θ	43°

$\theta = 44°$

Ans. $\begin{cases} \beta = 26° 46' 48'' \\ \omega = 54° 56' 45'' \end{cases}$

31. Determine the angle β.
32. Determine the angle ω.

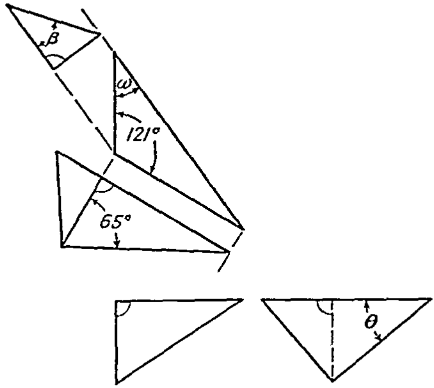


VARIABLE		
No.	Sym.	Value
1	θ	42°
2	θ	43°
3	θ	44°
4	θ	45°
5	θ	46°
6	θ	47°

$\theta = 48°$

Ans. $\begin{cases} \beta = 21° 33' 38'' \\ \omega = 30° 33' 46'' \end{cases}$

33. Determine the angle β.
34. Determine the angle ω.



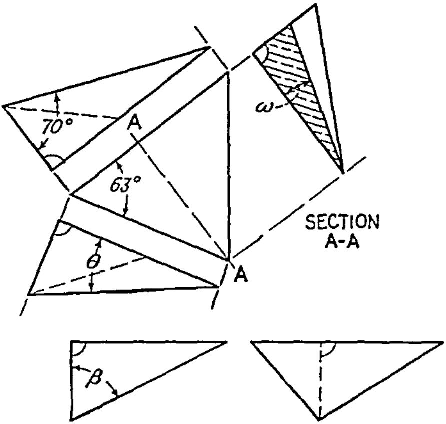
Variable		
No.	Sym.	Value
1	θ	37°
2	θ	38°
3	θ	39°
4	θ	40°
5	θ	41°
6	θ	42°

$\theta = 43°$

Ans. $\begin{cases} \omega = 40° \; 10' \; 43'' \\ \beta = 34° \; 40' \; 41'' \end{cases}$

35. Determine the angle ω.

36. Determine the angle β.



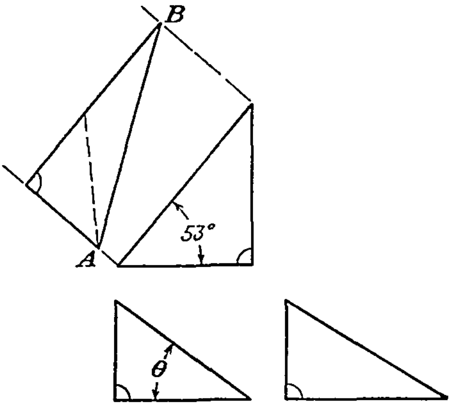
Variable		
No.	Sym.	Value
1	θ	25°
2	θ	26°
3	θ	27°
4	θ	28°
5	θ	29°
6	θ	30°

$\theta = 31°$

Ans. $\begin{cases} \beta = 58° \; 38' \; 15'' \\ \omega = 26° \; 3' \; 14'' \end{cases}$

37. Determine the angle β.

38. Determine the angle ω.



Variable		
No.	Sym.	Value
1	θ	36°
2	θ	37°
3	θ	38°
4	θ	39°
5	θ	40°
6	θ	41°

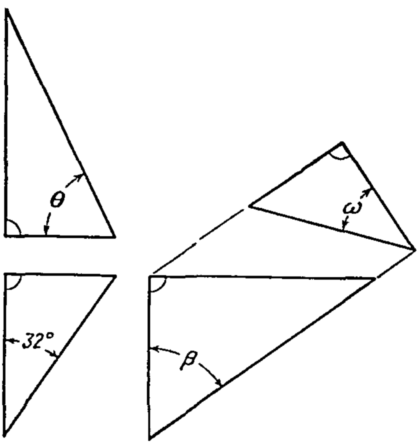
$\theta = 42°$

Ans. $\begin{cases} \beta = 28° \, 27' \, 8'' \\ \omega = 44° \, 36' \, 5'' \\ \Sigma = 31° \, 56' \, 46'' \end{cases}$

39. Determine the true angle β of line AB in reference to the horizontal plane.

40. Determine the true angle ω of line AB in reference to the vertical plane.

41. Determine the true angle Σ of line AB in reference to the profile plane.



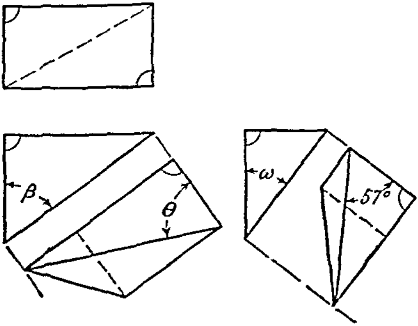
Variable		
No.	Sym.	Value
1	θ	63°
2	θ	64°
3	θ	65°
4	θ	66°
5	θ	67°
6	θ	68°

$\theta = 69°$

Ans. $\begin{cases} \beta = 58° \, 26' \, 13'' \\ \omega = 36° \, 15' \, 14'' \end{cases}$

42. Determine the angle β.

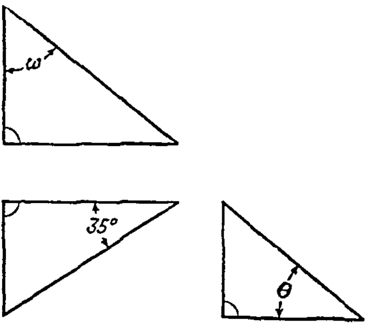
43. Determine the angle ω.



Variable		
No.	Sym.	Value
1	θ	66°
2	θ	67°
3	θ	68°
4	θ	69°
5	θ	70°
6	θ	71°

$$\theta = 72^\circ$$
$$Ans. \begin{cases} \beta = 34^\circ\ 56'\ 11'' \\ \omega = 21^\circ\ 37'\ 14'' \end{cases}$$

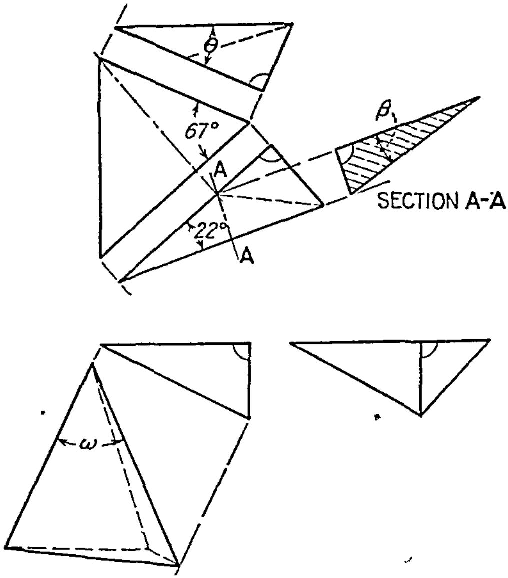
44. Determine the angle β.
45. Determine the angle ω.



Variable		
No.	Sym.	Value
1	θ	38°
2	θ	39°
3	θ	40°
4	θ	41°
5	θ	42°
6	θ	43°

$$\theta = 44^\circ$$
$$Ans.\ \omega = 54^\circ\ 3'\ 15''$$

46. Determine the angle ω.



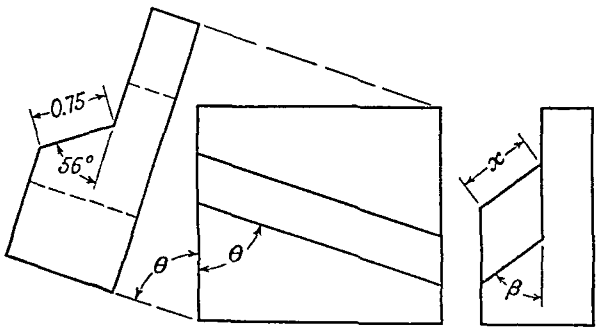
VARIABLE

No.	Sym.	Value
1	θ	25°
2	θ	26°
3	θ	27°
4	θ	28°
5	θ	29°
6	θ	30°

$\theta = 31°$

Ans. $\begin{cases} \beta = 24°\ 2'\ 48' \\ \omega = 44°\ 45'\ 12'' \end{cases}$

47. Determine the angle β.
48. Determine the angle ω.



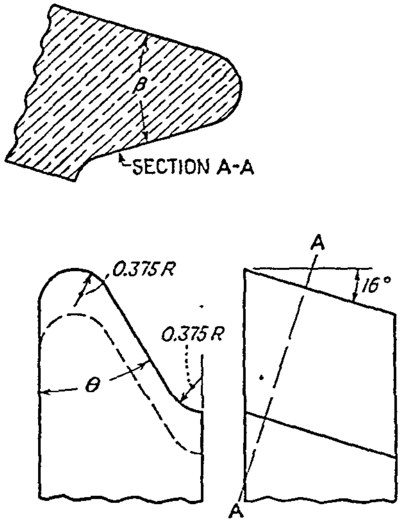
$\theta = 77°$

Ans. $\begin{cases} \beta = 55°\ 18'\ 25'' \\ x = .75622 \end{cases}$

VARIABLE

1. $\theta = 71°$ 2. $\theta = 72°$ 3. $\theta = 73°$
4. $\theta = 74°$ 5. $\theta = 75°$ 6. $\theta = 76°$

49. Determine the angle β.
50. Determine the distance x.

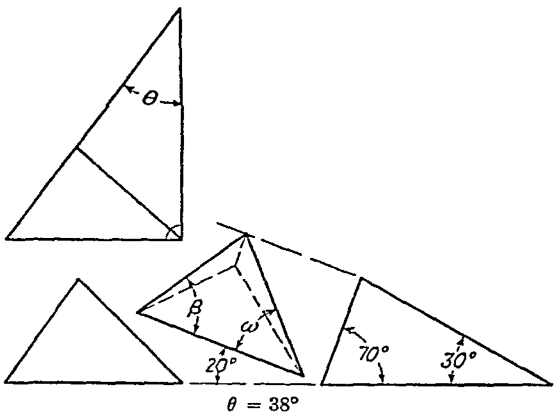


Variable		
No.	Sym.	Value
1	θ	29°
2	θ	30°
3	θ	31°
4	θ	32°
5	θ	33°
6	θ	34°

$\theta = 35°$

Ans. $\beta = 36° 4' 14''$

51. Determine the angle β.



$\theta = 38°$

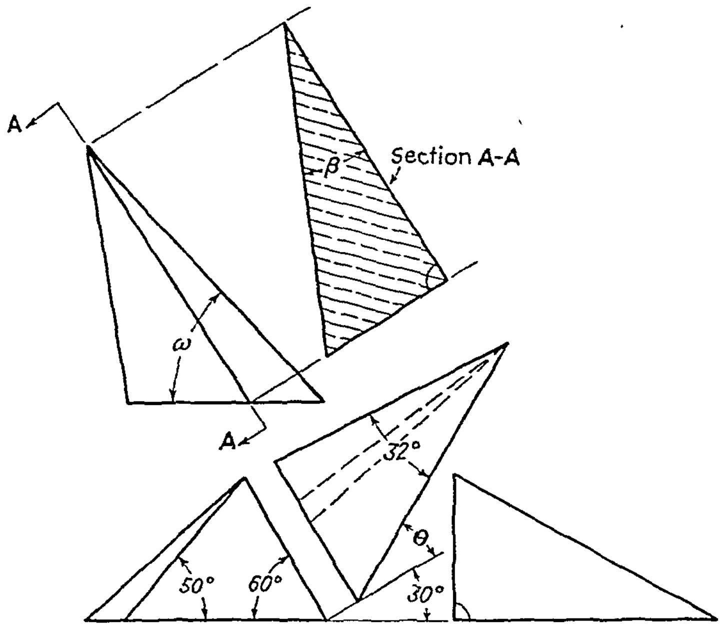
Ans. $\begin{cases} \beta = 75° 2' 21'' \\ \omega = 38° 10' 53'' \end{cases}$

Variable

1. $\theta = 32°$	**2.** $\theta = 33°$	**3.** $\theta = 34°$
4. $\theta = 35°$	**5.** $\theta = 36°$	**6.** $\theta = 37°$

52. Determine the angle β.

53. Determine the angle ω.



$$\theta = 28°$$
$$Ans. \begin{cases} \beta = 22° \ 30' \ 38'' \\ \omega = 43° \ 14' \ 23'' \end{cases}$$

VARIABLE

1. $\theta = 22°$	2. $\theta = 23°$	3. $\theta = 24°$
4. $\theta] = 25°$	5. $\theta = 26°$	6. $\theta = 27°$

54. Determine the angle β.
55. Determine the angle ω.

COMPOUND-ANGULAR HOLE BORING

In some of the practical compound-angle problems sufficient data for machining purposes may be given on blueprint in just two of the conventional planes. They may be given in either of the two following sets of planes, horizontal and vertical or vertical and profile. A typical example of a problem of this nature is the boring of a compound-angular hole. In boring a compound-angular hole, there are two important machining factors that must be considered. The mechanic, before he can proceed with the setting up of his job, must have the true angle of the hole in terms of some specified plane, together with the angle of rotation in respect to some plane of reference.

These two angles or either their complements or supplements, depending upon the nature of the job, are the angles at which the part to be machined must be set up on the machine in order that the hole be properly bored. The following is an illustrative problem; the given angles are α and Σ.

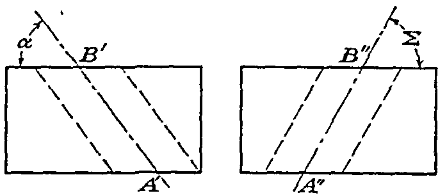


FIG. 41.

To draw this compound angular hole problem into a pictorial form proceed according to the rules and information given in the preceding problems with two exceptions: first, omit the view in the horizontal plane, since it was omitted in the original problem; second, draw a right triangle in each of the two given planes where the axis of the hole is the hypotenuse of each of the triangles. See the following illustration:

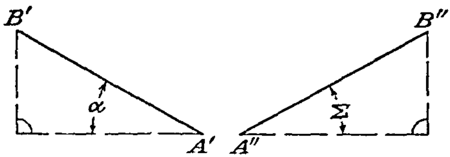


FIG. 42.

The mechanical drawing as shown in Fig. 42 without the horizontal view is ambiguous and represents two distinct objects with entirely different shapes. In order to draw the solid figure pertaining to a compound-angular hole, the student must bear in mind that the original problem was an angular-hole problem; therefore the axis of the angular hole must be shown in the pictorial drawing. Proceed by projecting the triangles in these two planes into the pictorial form as shown in Fig. 43. Be sure to draw the line AB, which represents the axis of the hole, before drawing any other line in the

pictorial view. The line AB is drawn from the intersection of the two projection lines $A''D$ and $A'E$ to the intersection of the two projection lines $B''B$ and $B'B$.

Now the pictorial drawing $ADCE$-B as shown in Fig. 43 is a rectangular pyramid made up of five faces, four of which are right triangles and the fifth of which is considered the base with a rectangular shape. Consequently, it is a Type III problem. Generally the compound-angular hole problems are of a Type III nature. Draw the line AC, which is the projection of the line AB into the lower horizontal plane. The

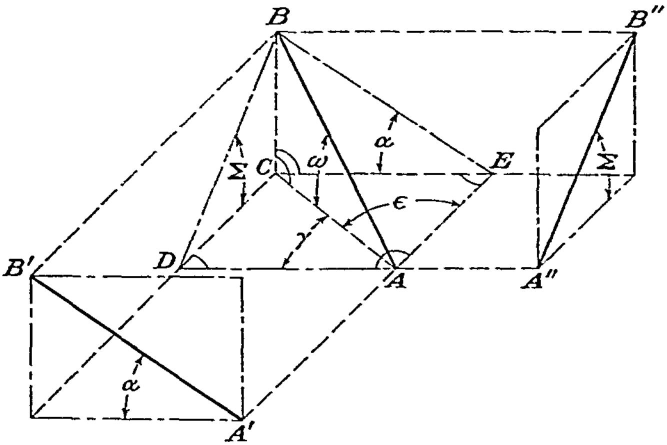


FIG. 43.

plane ABC divides the Type III solid figure into two Type I solid figures. The two Type I solid figures are the triangular pyramids ACD-B and ACE-B. The angles CAD (γ) and CAE (ϵ) are complementary angles and are the angles of rotation in reference to the vertical and profile planes, respectively. For practical purposes it is only necessary to compute one of these angles of rotation. The angle to be computed depends upon the manner in which the compound-angle job is set up on the machine. The angle of rotation and the true angle may now be computed in the usual Type III and Type I procedures. The angle of rotation (γ or ϵ) is computed with the Type III solid figure, and the true angle (ω) is computed with the Type I solid figure.

CHECKING THE LOCATION OF COMPOUND-ANGULAR HOLES

Let it be required to check the location of the compound-angular hole in terms of the vertical and profile planes.

The following is an illustrative problem and, in general, is typical of the manner in which a compound-angular hole is presented on blueprint to the mechanic for machining purposes.

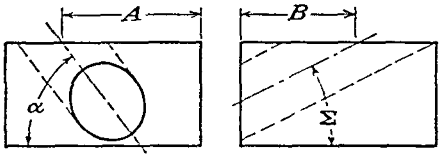


FIG. 44.

After the compound-angular hole has been properly set up according to the true angle and the angle of rotation, the location of the hole may be checked at different intervals with a **ball-plug gage.** The location of the hole is checked repeatedly until the **boring axis** of the hole is finally brought into its rightful place. The checking of the distance of the hole over the ball-plug gage in terms of its vertical and profile planes may be accomplished in many ways. Figures 45 and 46 show a method of procedure in which the checking of the distance over the ball-plug gage to the vertical plane and over the ball-plug gage to the profile plane may be accomplished with the aid of a parallel bar and a pair of micrometers.

Figure 45 shows the distance being checked from a parallel bar, which is on the surface of the vertical plane of the object, over the ball-plug gage.

Figure 46 shows the distance being checked from a parallel bar, which is on the surface of the profile plane of the object, over the ball-plug gage.

The ball-plug gage must be rigidly placed in the compound-angular hole, and then the height H, as shown in Fig. 47, is checked with an **indicator gage.** For accuracy this distance must be checked within a thousandth of an inch. When checking the distances x and y with a pair of micrometers, extra precaution must be taken in order not to disturb the height H of the ball-plug gage.

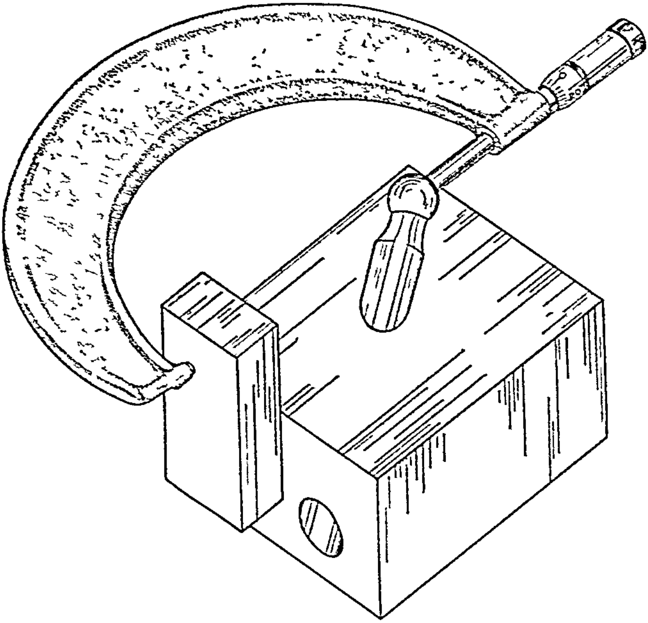


FIG 45

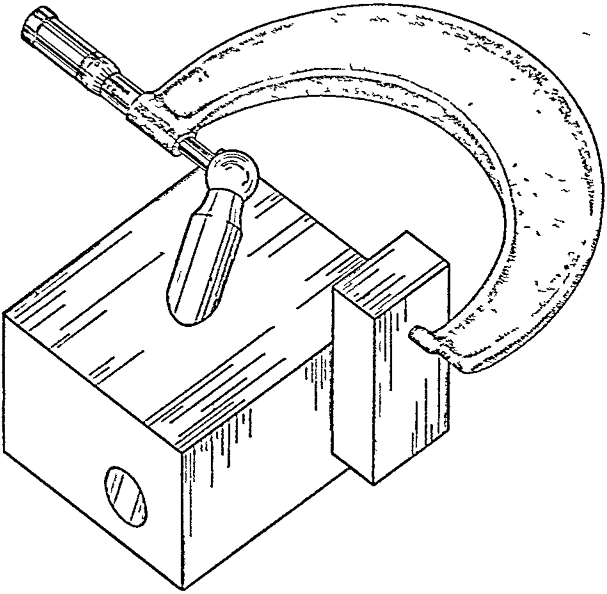


FIG 46.

If it is desired to check the location of the compound-angular hole with other surfaces, the procedure is practically the same as the one given below.

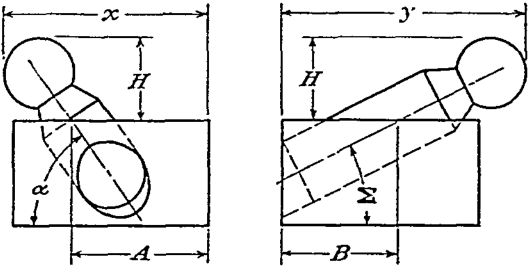


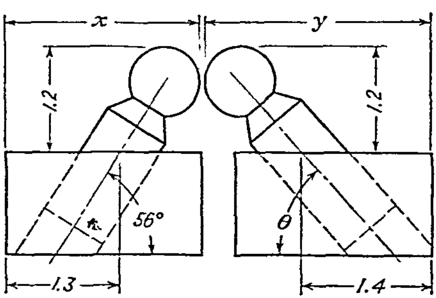
FIG. 47.

R = radius of the ball-plug gage,

$X = (H - R) \cot \alpha + A + R, \qquad Y = (H - R) \cot \Sigma + B + R.$

PRACTICAL COMPOUND-ANGULAR HOLE PROBLEMS

The following sixteen compound-angular hole problems have been drawn in a mechanical drawing form. They are to be drawn in a pictorial drawing form according to Fig. 43 on page 61 and then computed for the dimensions and angles as indicated in the original drawing.



VARIABLE		
No.	Sym.	Value
1	θ	47°
2	θ	48°
3	θ	49°
4	θ	50°
5	θ	51°
6	θ	52°

Radius of ball-plug gage = .3125

$\theta = 53°$

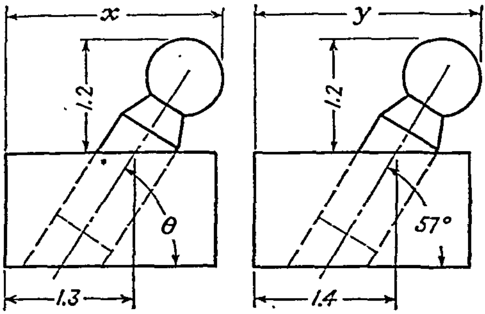
$$Ans. \begin{cases} \omega = 44° \, 40' \, 9'' \\ \epsilon = 41° \, 50' \, 55'' \\ x = 2.2111 \\ y = 2.3812 \end{cases}$$

56. Determine the true angle ω of the axis of the compound-angular hole in terms of the horizontal plane.

57. Determine the angle of rotation ϵ of the compound-angular hole in terms of the profile plane.

58. Determine the distance x.

59. Determine the distance y.



VARIABLE		
No.	Sym.	Value
1	θ	55°
2	θ	56°
3	θ	57°
4	θ	58°
5	θ	59°
6	θ	60°

Radius of ball-plug gage = .4375

$\theta = 61°$

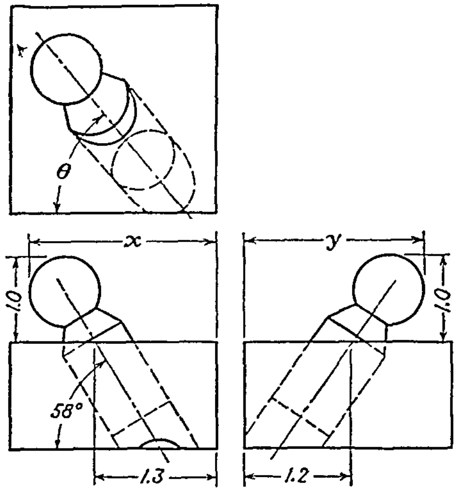
$$Ans. \begin{cases} \omega = 49° \, 30' \, 30'' \\ \gamma = 49° \, 31' \, 2'' \\ x = 2.1601 \\ y = 2.3326 \end{cases}$$

60. Determine the true angle ω of the axis of the compound-angular hole in terms of the horizontal plane.

61. Determine the angle of rotation γ of the compound-angular hole in terms of the vertical plane.

62. Determine the distance x.

63. Determine the distance y.



VARIABLE		
No.	Sym.	Value
1	θ	47°
2	θ	48°
3	θ	49°
4	θ	50°
5	θ	51°
6	θ	52°

Radius of ball-plug gage = .375

$\theta = 53°$

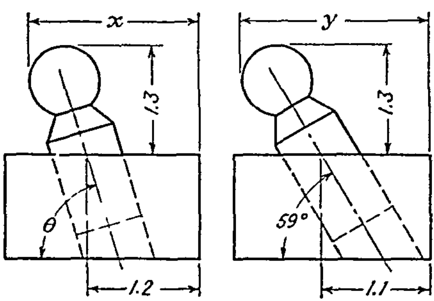
Ans. $\begin{cases} \omega = 43° \, 55' \, 24'' \\ \gamma = 53° \\ x = 2.0655 \\ y = 2.0932 \end{cases}$

64. Determine the true angle ω of the axis of the compound-angular hole in terms of the horizontal plane.

65. Determine the angle of rotation γ of the compound-angular hole in terms of the vertical plane.

66. Determine the distance x.

67. Determine the distance y.



VARIABLE		
No.	Sym.	Value
1	θ	77°
2	θ	78°
3	θ	79°
4	θ	80°
5	θ	81°
6	θ	82°

Radius of ball-plug gage = .250

$\theta = 83°$

Ans. $\begin{cases} \omega = 58° \, 28' \, 48'' \\ \epsilon = 11° \, 32' \, 57'' \\ x = 1.5789 \\ y = 1.9808 \end{cases}$

68. Determine the true angle ω of the axis of the compound-angular hole in terms of the horizontal plane.

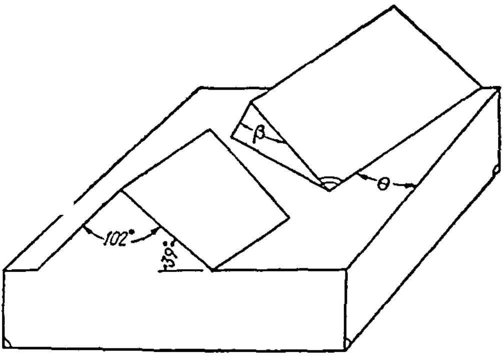
69. Determine the angle of rotation ϵ of the compound-angular hole in terms of the profile plane.

70. Determine the distance x.

71. Determine the distance y.

PRACTICAL COMPOUND-ANGLE PROBLEMS

The next group of problems are practical shop and drawing room compound-angle problems. The students must recognize which of the five types are involved and draw whatever construction lines are necessary to complete the figure of that type. It is quite evident from the preceding discussions that the procedure in forming a solution for all types is similar to that given for Type I on page 11 and should be generally followed. In some of these problems, diagrammatic hints or final formulas are given. The student is again urged to make use of plastic clay.

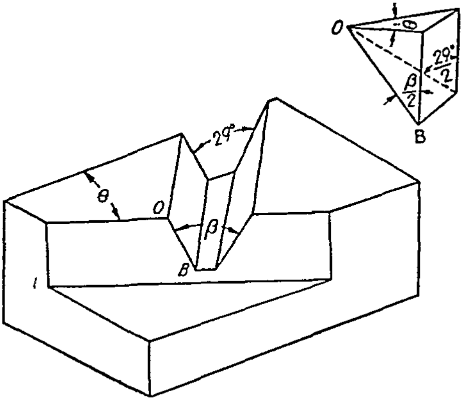


Variable		
No.	Sym.	Value
1	θ	16°
2	θ	18°
3	θ	20°
4	θ	22°
5	θ	24°
6	θ	26°

$\theta = 28°$

Ans. $\beta = 94° 56' 52''$

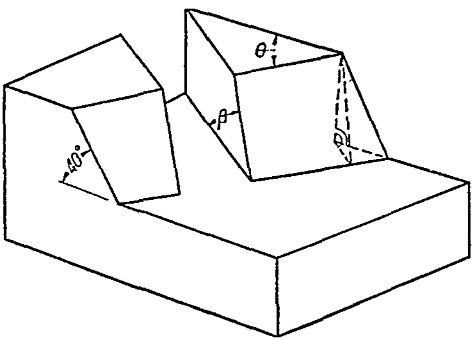
1. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	18°
2	θ	21°
3	θ	24°
4	θ	27°
5	θ	30°
6	θ	33°

$\theta = 15°$
Ans. $\beta = 28° \ 3' \ 0''$

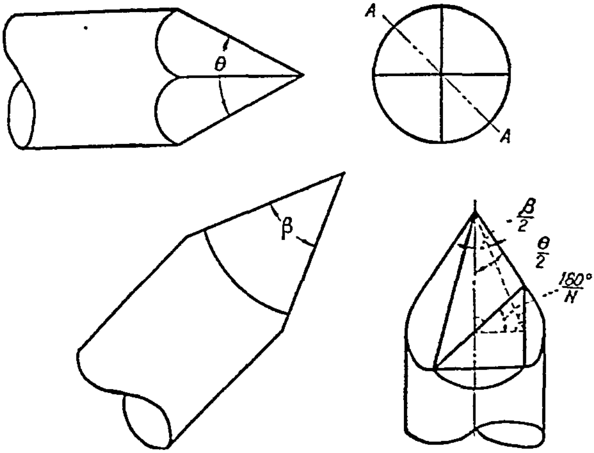
2. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	47°
2	θ	51°
3	θ	55°
4	θ	59°
5	θ	63°
6	θ	67°

$\theta = 43°$
Ans. $\beta = 50° \ 53' \ 45''$

3. Determine the angle β.

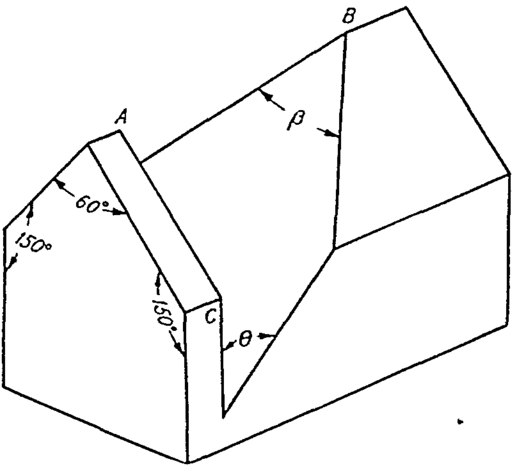


Diagrammatic Hint

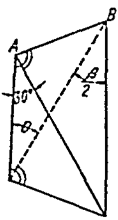
VARIABLE		
No.	Sym.	Value
1	θ	72°
2	θ	73°
3	θ	74°
4	θ	75°
5	θ	76°
6	θ	77°

$\theta = 71°$
Ans. $\beta = 53° \ 32' \ 0''$
N = number of sides

4. Determine the angle β.



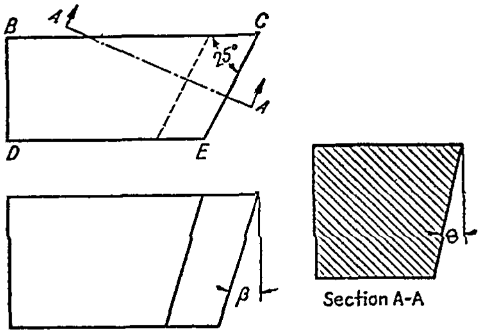
VARIABLE		
No.	Sym.	Value
1	θ	22°
2	θ	23°
3	θ	24°
4	θ	25°
5	θ	26°
6	θ	27°



Diagrammatic Hint

$\theta = 21°$
Ans. $\beta = 56° \ 39' \ 4''$

5. Determine the angle β.



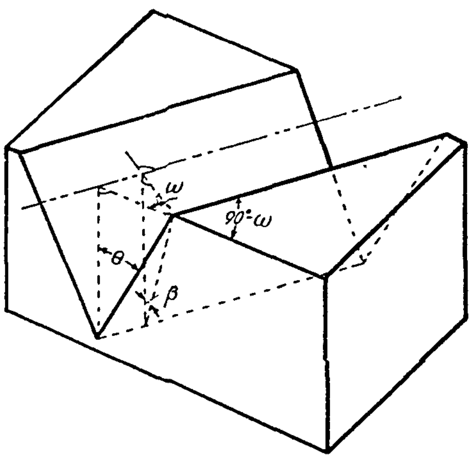
Variable		
No.	Sym.	Value
1	θ	9°
2	θ	10°
3	θ	11°
4	θ	12°
5	θ	13°
6	θ	14°

$\theta = 8°$
Ans. $\beta = 18° \, 23' \, 39''$

In the mechanical drawing above, the angle θ shown in the section *A-A* is the clearance of the face *BCDE* measured from a line perpendicular to the top surface and the cutting edge *CE*.

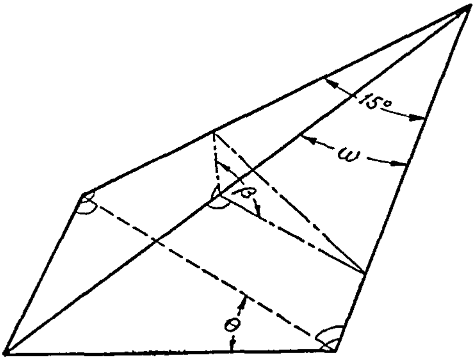
6. Determine the angle β.

Verify: $\tan \beta = \csc 25° \tan \theta$.



Given	To find	Formula
7. θ and ω.......	β	$\tan \beta = \cos \omega \tan \theta$
8. θ and β.......	ω	$\cos \omega = \tan \beta \cot \theta$

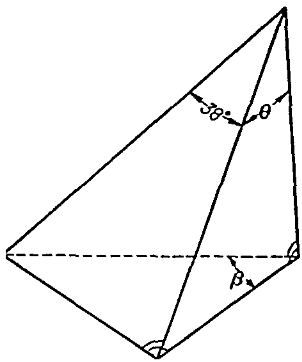
Name the type of problem to be used in the foregoing two problems, and rify the two formulas.



VARIABLE		
No.	Sym.	Value
1	θ	20°
2	θ	24°
3	θ	28°
4	θ	32°
5	θ	36°
6	θ	40°

$\theta = 44°$
Ans. $\beta = 47° \, 51' \, 22''$

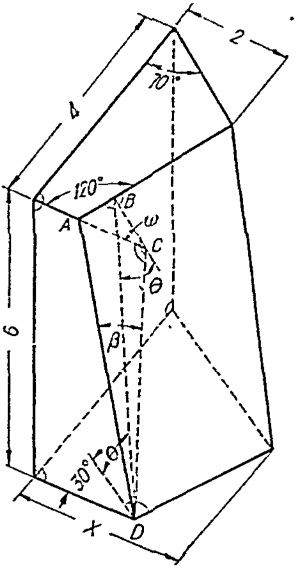
9. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	15°
2	θ	17°
3	θ	19°
4	θ	21°
5	θ	23°
6	θ	25°

$\theta = 27°$
Ans. $\beta = 59° \, 50' \, 26''$

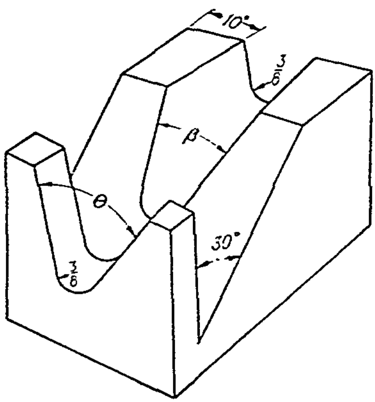
10. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	65°
2	θ	67°
3	θ	69°
4	θ	71°
5	θ	73°
6	θ	75°

$\theta = 77°$
Ans. $x = 3.3218$

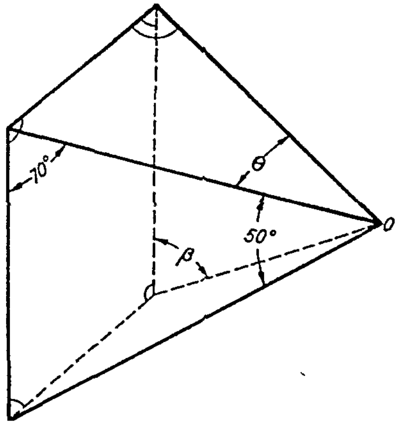
20. Determine the distance x.



VARIABLE		
No.	Sym.	Value
1	θ	38°
2	θ	40°
3	θ	42°
4	θ	44°
5	θ	46°
6	θ	48°

$\theta = 50°$
Ans. $\beta = 35° 2' 18''$

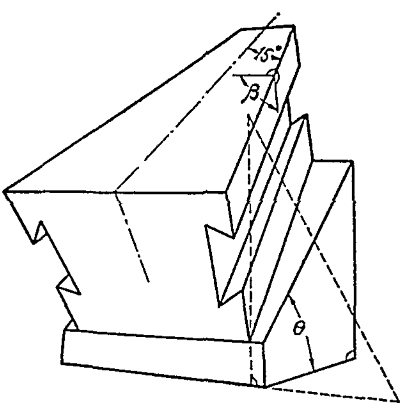
Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	20°
2	θ	22°
3	θ	24°
4	θ	26°
5	θ	28°
6	θ	30°

$\theta = 32°$
Ans. $\beta = 55° \ 2' \ 20''$

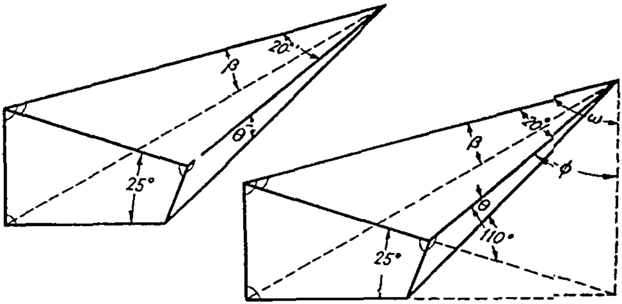
22. Determine the angle β.
Hint: Pass a plane through O perpendicular to the rectangular p



VARIABLE		
No.	Sym.	Value
1	θ	28°
2	θ	30°
3	θ	32°
4	θ	34°
5	θ	36°
6	θ	38°

$\theta = 40°$
Ans. $\beta = 77° \ 44' \ 49''$

23. Determine the angle β.

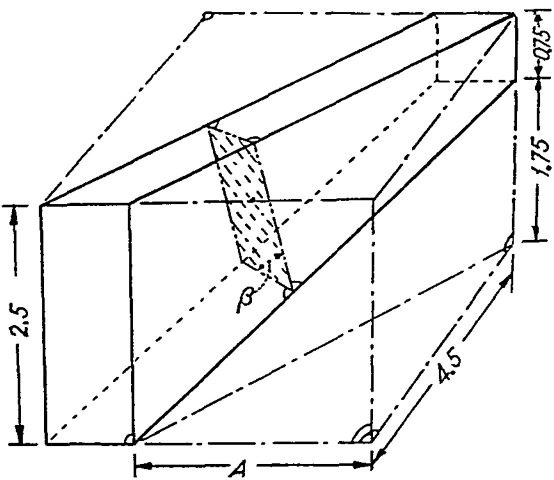


$\theta = 16°$
Ans. $\beta = 23° \ 17' \ 13''$

VARIABLE

1. $\theta = 4°$	2. $\theta = 6°$	3. $\theta = 8°$
4. $\theta = 10°$	5. $\theta = 12°$	6. $\theta = 14°$

24. Determine the angle β.
The diagram at the right indicates the method of solution.

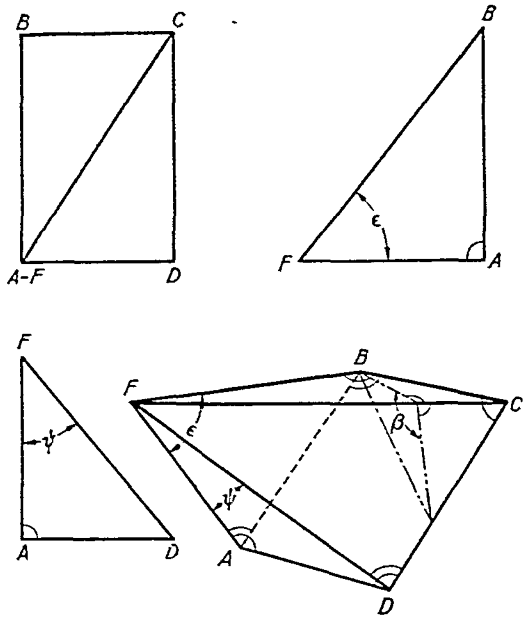


$A = 3$
Ans. $\beta = 78° \ 24'$

VARIABLE

1. $A = 2.25$	2. $A = 2.375$	3. $A = 2.5$
4. $A = 2.625$	5. $A = 2.75$	6. $A = 2.875$

25. Determine the angle β.



$\psi = 27°,\ \epsilon = 39°$

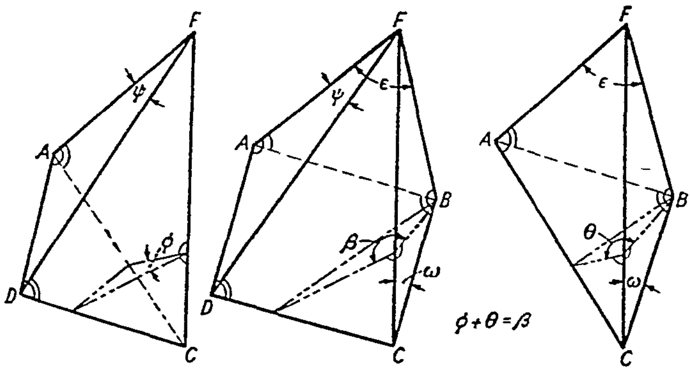
Ans. $\beta = 106°\ 36'\ 2''$

DOUBLE VARIABLE

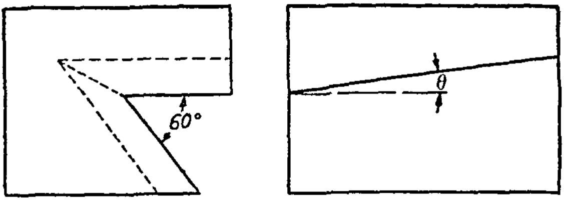
No.	Sym.	Value	Sym.	Value	No.	Sym.	Value	Sym.	Value
1	ψ	39°	ϵ	51°	2	ψ	37°	ϵ	49°
3	ψ	35°	ϵ	47°	4	ψ	33°	ϵ	45°
5	ψ	31°	ϵ	43°	6	ψ	29°	ϵ	41°

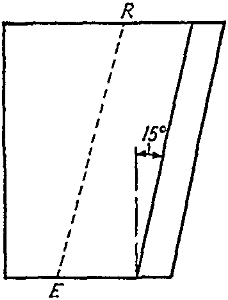
26. Determine the angle β.

Hint: The diagrams below show how this figure may be split into two parts giving two figures each of Type I.



$\cos \Sigma = \sin \psi \sin \epsilon$ $\qquad\qquad$ $\beta = 180° - \Sigma$





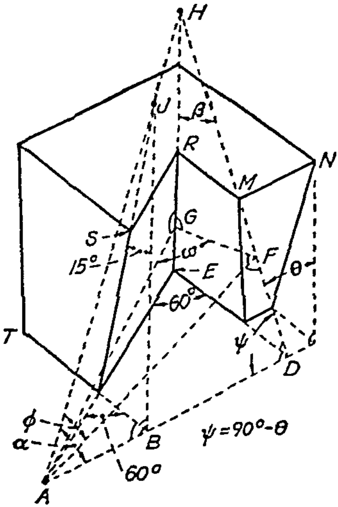
$\theta = 22°$

Ans. $\omega = 68° \, 28' \, 35''$

VARIABLE

1. $\theta = 10°$ 2. $\theta = 12°$ 3. $\theta = 14°$
4. $\theta = 16°$ 5. $\theta = 18°$ 6. $\theta = 20°$

27. Determine the projection of the angle 60° in a plane perpendicular to the line RE, which is the angle ω in the following figure.

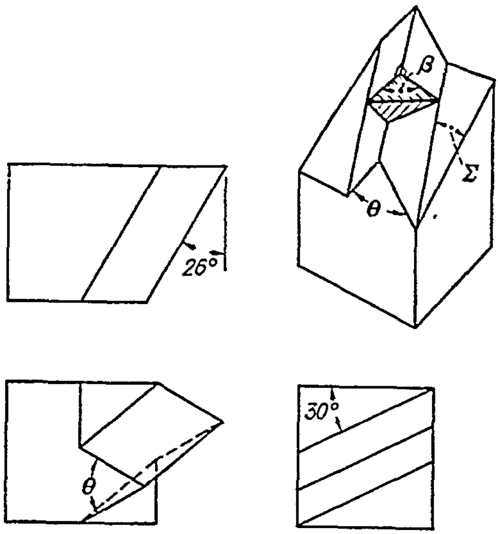


olution on next page.)

Solution of Preceding Problem: Through MN pass a plane perpendicular to the lower base (extended). Produce the lines ER and DM until they meet (at H). Through D draw a line, in the lower base extended, parallel to MN which will meet EC produced at A. Connect points A and H. Thus a pyramid ADH-E of Type V is formed. The plane of the face TCS cuts the line AD at B and the line AH at J. The pyramid ABJ-C is of Type II with angles BJC and BCA given (15° and 60°, respectively). Determine the angle α. Now use pyramid ADH-E (Type V), having angles α, ψ, and 60° given, to determine the angle β.

Through A draw a plane perpendicular to EH meeting EH at G and meeting HD at F. This gives the pyramid AGH-F which is of Type I, having the given angles β and ϕ (where $\phi = \alpha - \theta$).

Determine the angle ω which is the required projection of the angle 60° on the plane perpendicular to the line RE.



$\theta = 62°$

Ans. $\beta = 60° \, 8' \, 8''$

VARIABLE

1. $\theta = 56°$ 2. $\theta = 57°$ 3. $\theta = 58°$
4. $\theta = 59°$ 5. $\theta = 60°$ 6. $\theta = 61°$

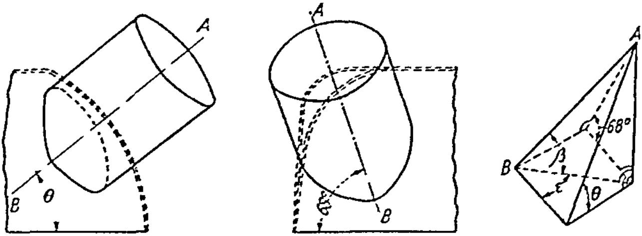
28. Determine the angle β.

$$\tan \Sigma = \tan 30° \cos 26°,$$

$$\tan \phi = \tan 26° \cos 30°, \qquad \sin \phi = \sin 26° \cos \Sigma,$$

$$\tan \frac{\beta}{2} = \tan \frac{\theta}{2} \cos \Sigma \sec \phi.$$

Fuel Filler Spout



Diagrammatic Hint

$$\theta = 46^\circ$$

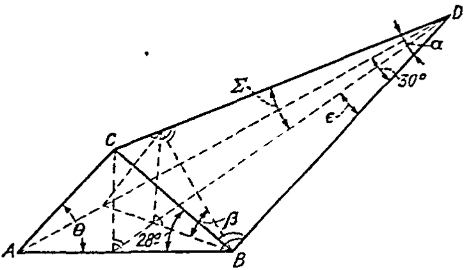
$$Ans. \begin{cases} \beta = 43^\circ\ 41'\ 20'' \\ \omega = 67^\circ\ 17'\ 45'' \end{cases}$$

Variable

1. $\theta = 34^\circ$	2. $\theta = 36^\circ$	3. $\theta = 38^\circ$
4. $\theta = 40^\circ$	5. $\theta = 42^\circ$	6. $\theta = 44^\circ$

29. Determine the true angle β of the axis of the spout with the base.

30. Determine the angle of rotation ω which is the angle made by the profile plane and the projection of the spout on the horizontal plane.



Variable

No.	Sym.	Value
1	θ	30°
2	θ	32°
3	θ	34°
4	θ	36°
5	θ	38°
6	θ	40°

$$\theta = 42^\circ$$

$$Ans.\ \beta = 26^\circ\ 11'\ 7''$$

31. Determine the angle β which is in a plane perpendicular to the line CD.

The angle α must be computed before the angle β can be determined. The solution for computing the angle α is as follows:

1. Special type.
2. AB is common to the two complete triangles ABC and ABD Therefore, let $AB = 1$.
3. $BC = \dfrac{\csc 28^\circ}{\cot \theta + \cot 28^\circ}$, $\qquad BD = \cot 30^\circ$.
4. $\cot \alpha = \dfrac{BD}{BC} = \dfrac{\cot 30^\circ}{\dfrac{\csc 28^\circ}{\cot \theta + \cot 28^\circ}} = \dfrac{\cot 30^\circ(\cot \theta + \cot 28^\circ)}{\csc 28^\circ}.$

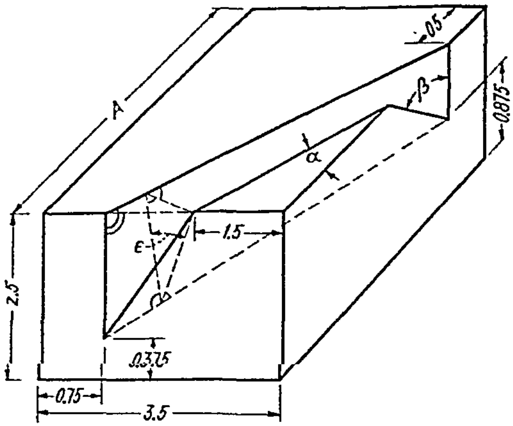
5. $\cot \alpha = \sin 28° \cot 30° (\cot \theta + \cot 28°)$.

The student should verify the following three formulas:

$\cot \epsilon = \cot \alpha \sec 28°$.

$\sin \Sigma = \sin 28° \sin \alpha$.

$\sin \beta = \tan \Sigma \cot \alpha$.

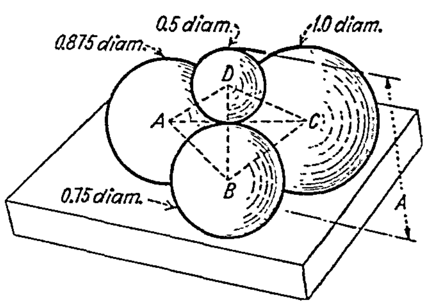


Variable		
No.	Sym.	Value
1	A	5.25
2	A	5.375
3	A	5.5
4	A	5.625
5	A	5.75
6	A	5.875

$A = 5.125$

Ans. $\begin{cases} \beta = 47° 55' 37'' \\ \alpha = 27° 58' 5'' \\ \epsilon = 27° 34' 55'' \end{cases}$

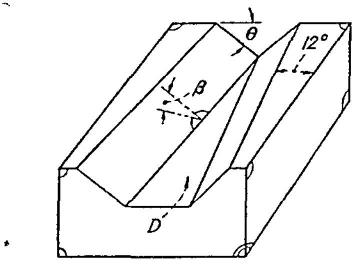
32. Determine the angle β.
33. Determine the angle α.
34. Determine the angle ϵ.



Ans. $A = 1.1391$

Given three balls on a surface plate tangent to each other and a fourth ball lying on top and touching each of the other three.

35. Determine the distance A.

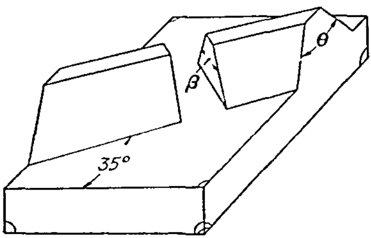


VARIABLE

No.	Sym.	Value
1	θ	35°
2	θ	37°
3	θ	39°
4	θ	41°
5	θ	43°
6	θ	45°

$\theta = 47°$
Ans. $\beta = 47° \, 37' \, 50''$
The face D is parallel to the base of the block

36. Determine the angle β.

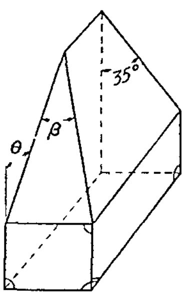


$\theta = 74°$
Ans. $\beta = 63° \, 22' \, 18''$

VARIABLE

1. $\theta = 62°$	**2.** $\theta = 64°$	**3.** $\theta = 66°$
4. $\theta = 68°$	**5.** $\theta = 70°$	**6.** $\theta = 72°$

37. Determine the angle β.

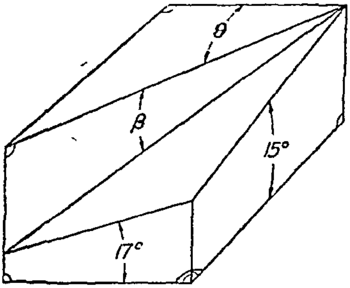


VARIABLE

No.	Sym.	Value
1	θ	18°
2	θ	20°
3	θ	22°
4	θ	24°
5	θ	26°
6	θ	28°

$\theta = 30°$
Ans. $\beta = 31° \, 13' \, 57''$

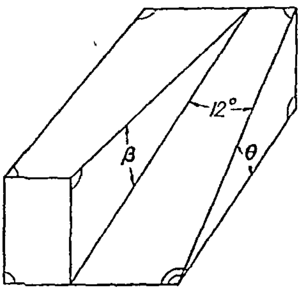
38. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	θ	60°
2	θ	62°
3	θ	64°
4	θ	66°
5	θ	68°
6	θ	70°

$\theta = 72°$
Ans. $\beta = 19° \; 15' \; 18''$

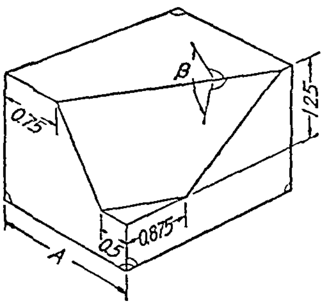
39. Determine the angle β.
Verify $\tan \beta = \sin \theta \tan 15° + \cos \theta \tan 17°$.



VARIABLE		
No.	Sym.	Value
1	θ	25°
2	θ	27°
3	θ	29°
4	θ	31°
5	θ	33°
6	θ	35°

$\theta = 37°$
Ans. $\beta = 36° \; 3' \; 43''$

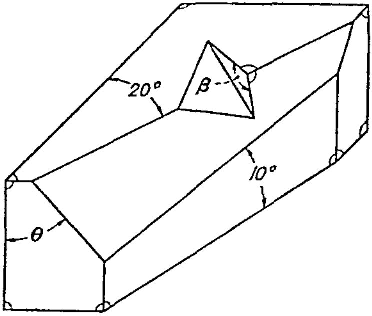
40. Determine the angle β.



VARIABLE		
No.	Sym.	Value
1	A	2.000
2	A	2.125
3	A	2.250
4	A	2.375
5	A	2.500
6	A	2.625

$A = 2.750$
Ans. $\beta = 136° \; 10' \; 30''$

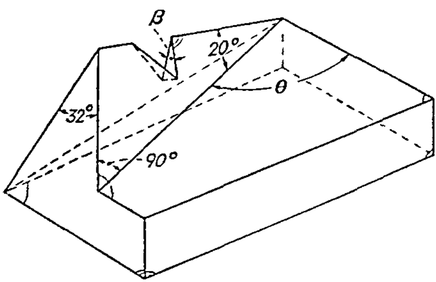
41. Determine the angle β.



Variable		
1	θ	46°
2	θ	48°
3	θ	50°
4	θ	52°
5	θ	54°
6	θ	56°

$\theta = 58°$
Ans. $\beta = 146° \, 22' \, 37''$

42. Determine the angle β.

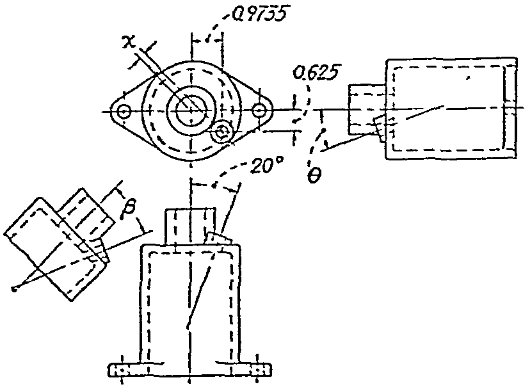


$\theta = 58°$
Ans. $\beta = 32° \, 40' \, 6''$

Variable

1. $\theta = 70°$ **2.** $\theta = 68°$ **3.** $\theta = 66°$
4. $\theta = 64°$ **5.** $\theta = 62°$ **6.** $\theta = 60°$

43. Determine the angle β.



VARIABLE

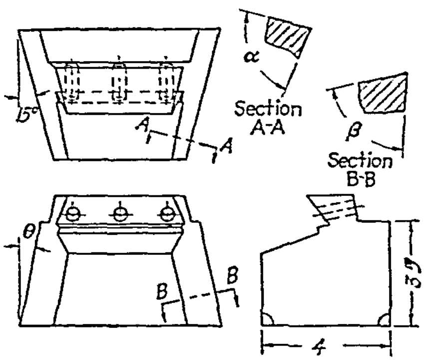
No.	Sym.	Value
1	θ	16°
2	θ	18°
3	θ	20°
4	θ	22°
5	θ	24°
6	θ	26°

$$\theta = 28^\circ$$

$$Ans. \begin{cases} \beta = 32^\circ\, 47'\, 43'' \\ x = .45028 \end{cases}$$

44. Determine the angle β.
45. Determine the distance x.

Tool Holder



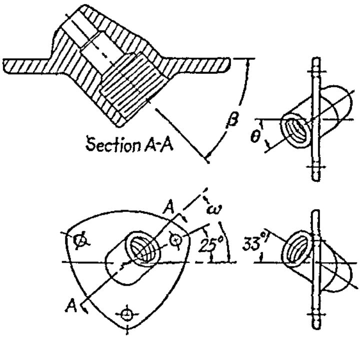
$$\theta = 20^\circ$$

$$Ans. \begin{cases} \alpha = 70^\circ\, 37'\, 47'' \\ \beta = 75^\circ\, 52'\, \ 2'' \end{cases}$$

VARIABLE

1. $\theta = 8^\circ$	2. $\theta = 10^\circ$	3. $\theta = 12^\circ$
4. $\theta = 14^\circ$	5. $\theta = 16^\circ$	6. $\theta = 18^\circ$

46. Determine the angle α.
47. Determine the angle β.

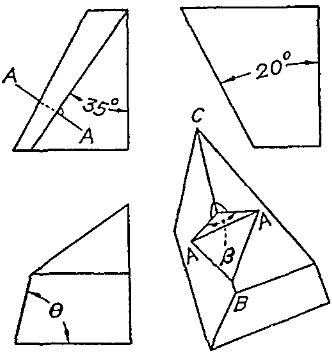


Variable		
No.	Sym.	Value
1	θ	24°
2	θ	26°
3	θ	28°
4	θ	30°
5	θ	32°
6	θ	34°

$\theta = 35°$

Ans. $\begin{cases} \omega = 42° \; 50' \; 36'' \\ \beta = 46° \; 19' \; 12'' \end{cases}$

48. Determine the angle ω.
49. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	65°
2	θ	66°
3	θ	67°
4	θ	68°
5	θ	69°
6	θ	70°

$\theta = 71°$

Ans. $\beta = 96°3'10''$

50. Determine the angle β.

CHAPTER II

COMPOUND ANGLES APPLIED TO THE MOUNTING OF PARTS ON ADJUSTABLE ANGLE PLATES

Occasionally a die part, jig part, or machine part has to be set up on a machine in a double or triple angular position. The part is put in this angular position in order that it may be properly machined. Generally, work of this nature is set up on the machine by a cut and try system, which is long and wasteful. The system is long because there is a repetition of work done by setting up the job at angles assumed to be right, then machining and checking. This process of setting the work up on assumed angles is repeated until a final position is obtained which will give a finished job having angles which are approximately correct. The system is wasteful because in many cases the part is scrapped on account of the piece being reset and remachined so often that it has finally become undersize.

The accurate setting up of any die section, jig part, or machine part in a double or triple angular position involves the determination of certain angles which can best be obtained by the use of compound angles. Hence the knowledge of compound angles is very important and necessary to tool makers, die makers, and engineers, for with this knowledge a great deal of scrapping and waste of time can be eliminated.

Tilting Angles and Angles of Rotation

In order properly to machine any part which requires double or triple angular settings, the part must be tilted and rotated through proper angles. The tilting is accomplished by placing the part on a plane, $AA'B'B$, called the **tilting plane,** which is inclined to the base $AA'C'C$ by an angle θ (Fig. 48) called the **tilting angle.**

In Fig. 48, it is evident that to obtain this tilting angle the plane on which the part is placed must be turned about a line AA' called the **tilting axis.**

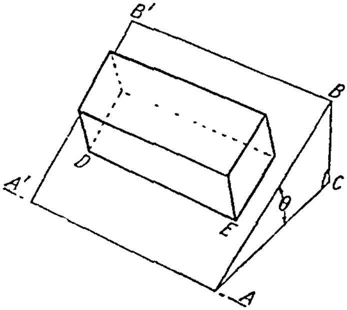


FIG. 48.

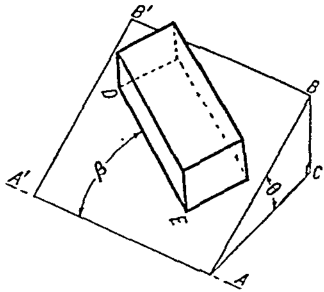


FIG. 49.

The part is shown in Fig. 48 with its base edges parallel to the tilting axis AA'. This part may now be rotated on the tilting plane so that the base edge DE will make an angle β (Fig. 49) with the tilting axis. This angle β is called the **angle of rotation** on the tilting plane.

MACHINING ANGULAR SURFACES PARALLEL TO BASE OF FIXTURE

Consider the die section shown below. The angles α and β are required on the finished work, and it is desired to machine the block so as to form a face $AFGH$ which will produce these

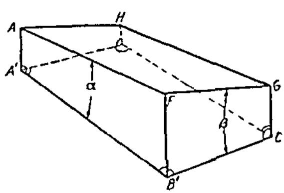


FIG. 50.

angles. To accomplish this, the block must be placed on the adjustable angle plate with the proper tilting angle and angle of rotation. The finished product is shown in the following figure and it will be noted that the base of the part has been placed

on the tilting plate and the top plane has been machined so that it is parallel to the base of the adjustable angle plate.

It is necessary to compute the proper angles of tilt and rotation (ω and θ) in order that this finished product may be obtained.

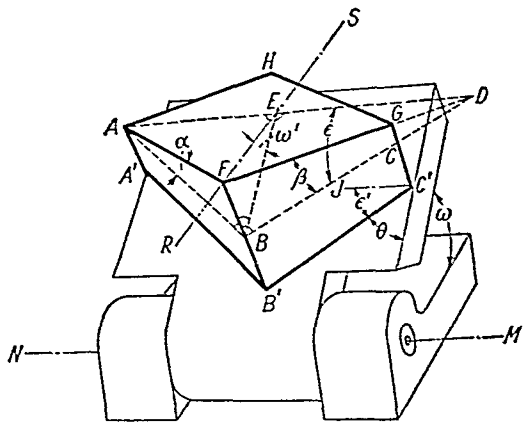


FIG. 51.

The plane ABC is constructed parallel to the base $A'B'C'$ and intersects the top plane in the line AD. The piece must be placed on the tilting plate so that this line AD is parallel to the tilting axis MN. This can be accomplished by rotating the block through the proper angle θ. Angle θ is the complement of angle ϵ. Angle ϵ' is formed by constructing $C'J$ in the base plane parallel to AD. Therefore angle ϵ' = angle ϵ by Proposition 51.

The pyramid ABF-D containing the given angles α and β, and the required angle ϵ is recognized as that of Type II and the angle ϵ may be obtained in the usual manner.

To compute the tilting angle ω, a plane is passed through FB perpendicular to AD. Angle ω' = angle ω. (EF is parallel to the base and perpendicular to AD at E. EB is parallel to the tilting plane and perpendicular to AD at E. Therefore angle ω' is equal to the angle between the two planes which by definition is ω.)

The pyramid *EFB-D* containing the known angles β and ϵ and the unknown angle ω', is recognized as that of Type I and angle ω' may be computed by the usual solution.

SETTING BLOCK ON FIXTURE FOR ANGULAR BORING

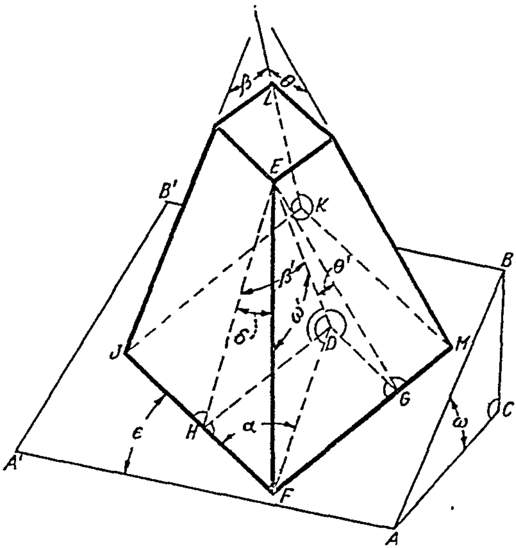


FIG. 52.

In another common type of problem it is necessary to place a block on the tilting plane in such a manner that a certain line *EF* be perpendicular to the base. This line *EF* is the intersection of two planes which make angles β and θ with mutually perpendicular planes of reference.

The angle between two intersecting lines drawn perpendicular, respectively, to two planes is equal to the angle between the two planes. Consider the line *EF* to be perpendicular to the base plane *A'AC* and the line *ED* perpendicular to the base of the die section which rests on the tilting plane. The plane *EFD* including these two perpendicular lines must be perpendicular to the tilting axis *A'A* in order that the angle ω' be equal to the tilting angle ω of the adjustable angle

plate. In order to fulfill this latter condition, it is necessary to determine the angle of rotation ϵ, which in this case may be obtained by computing its complementary angle α. Construct the plane EHD parallel to the face JKL, and the plane EGD parallel to the face MKL. The angles β' and θ' are equal, respectively, to β and θ. The figure $HFGD$-E is that of Type III. To compute angle α let ED be unity. Then $DH = \tan \beta'$ and $DG = HF = \tan \theta'$,

$$\cot \alpha = \frac{HF}{HD} = \frac{\tan \theta'}{\tan \beta'} = \tan \theta' \cot \beta'.$$

The pyramid FDH-E involving the angle ω' is of Type I where α and β' are given, and ω', which is equal to the tilting angle ω, may be computed in the usual manner.

DOUBLE ROTATION, SINGLE TILT AND DOUBLE TILT, SINGLE ROTATION

When only one element (as a plane or a line) is to be considered in setting a part on a machine, where the plane or the

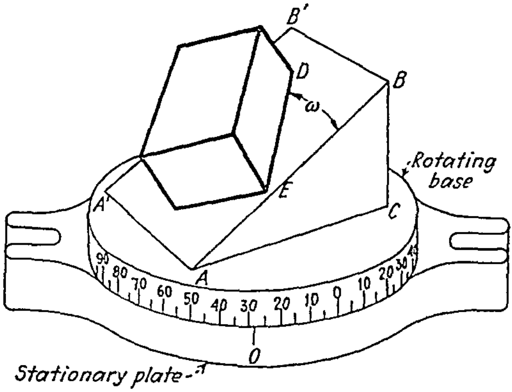


Fig. 53.

line must be either parallel or perpendicular to the base of the fixture, a single rotation and a single tilt will suffice.

But when two elements (as a plane and a line) must be considered together in setting a part on the machine, where the plane containing the line must be either parallel or perpendicular to the base of the fixture, and when the line in the plane must be set in a definite direction (as in setting the

line parallel to the path of a tool or a grinding wheel), a double rotation and single tilt, or a single rotation and a double tilt, is necessary.

The foregoing figure shows a device for accomplishing the double rotation and single tilt. The stationary plate is fastened to the table of the machine and the rotating base is rotated on the stationary base to any desired angle (approximately 28° in the figure).

The tilting block $AA'CBB'$ is fastened rigidly to the rotating plate and the work is fastened on the tilting plane so as to make the desired angle of rotation ω on the tilting plane $AA'B'B$.

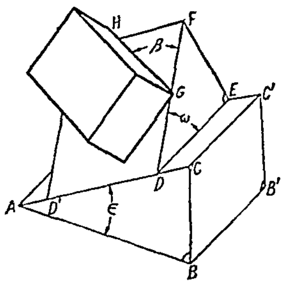


FIG. 54.

The general arrangement for accomplishing the single rotation and double tilt is shown in the figure above. Note that the two tilts are about axes which are at right angles to one another. Instead of using a special fixture for double tilting, one of the tilts may be accomplished by using the compound on the machine, such as the compound on a shaper or on a planer, or by using a special attachment which comes with the machine.

The same result on the finished product can be obtained by either of the foregoing arrangements, but the angles of tilt and rotation are different in the two cases and involve different solutions of compound angles. The mechanic must consider the equipment available in determining which method of solution must be used.

The same type of problem will now be worked by each of the foregoing methods. In the first solution, all of the steps will

be given and the student should carry out the numerical work. The second method is given in the form of a problem, which the student should solve with the aid of the diagrammatic hint.

Double-rotation, Single-tilt Method

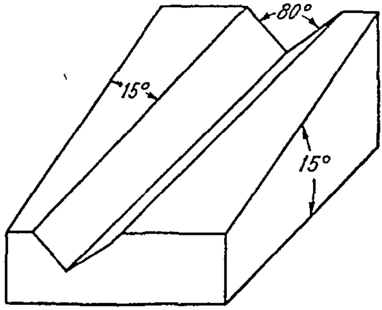


FIG. 55.

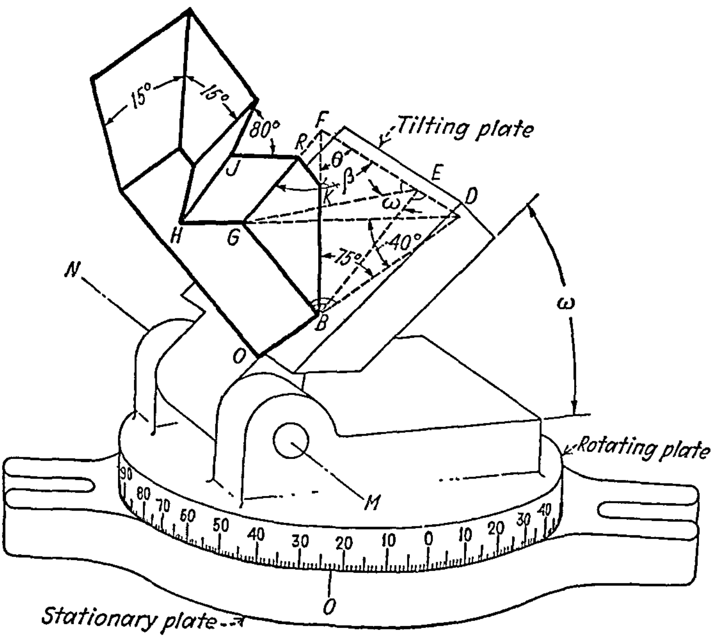


FIG. 56.

Example: Determine the angles of rotation which are β and θ in Fig. 56 and the angle of tilt which is ω.

(*Solution on next page.*)

(Solution continued from previous page.)

It is desired to form the angular V in the die section shown above. To do this, it is necessary to set up the die section on the tilting plane in such a manner that the face *HGJ* be parallel to the base of the adjustable angle plate, and the line *HJ* parallel to the path of the tool. In this case consider the path of the tool to be parallel to a line passing through the center and the zero mark of the stationary plate. Careful attention should be given to the path of the tool when two intersecting surfaces are to be machined, as it plays an important part in the setting up of such jobs and should always be considered before making any calculations.

Solution: The lines *HG* and *OB* are extended and meet in the point *D*. Similarly, *GR* and *BK* are extended and meet at *F*. Join *F* and *D*. This forms the pyramid *FBD-G*, which is of Type V, having the given angles *GDB* (40°) and *FBD* (75°) and *GFB* (15°). The construction line *FD* is parallel with the tilting axis *MN*, because, if two parallel planes are cut by a third plane, the two lines of intersection are parallel, thus making θ the angle of rotation on the tilting plate. Solve for the angle θ in the pyramid *FBD-G*.

Pass a plane through *GB* perpendicular to *FD* cutting *FD* at *E*. The angle *GEB* (ω) is the tilting angle.

The pyramid *FGB-E* is of Type I, having the known angles *GFB* (15°) and θ. Solve for the angles β and ω. Since *GF* (which is parallel to *HJ*) must be parallel to the zero reference line, and since the plane *GFD* is parallel to the base, the angle β is the angle of rotation on the rotating base (shown to be about 23° in the figure).

Single-rotation, Double-tilt Method

Figure 58 (next page) shows a die section similar to the one of the previous problem and Fig. 57 shows it properly mounted for machining with plane *EF* parallel to plane *AC*, and *DF* and *EG* parallel to *BC*. In Fig. 59, construction lines have been drawn and the angles of tilt and rotation labeled to indicate to the student the solution of the problem. The student should recognize the various type figures involved and carry out the solution as in the previous problem.

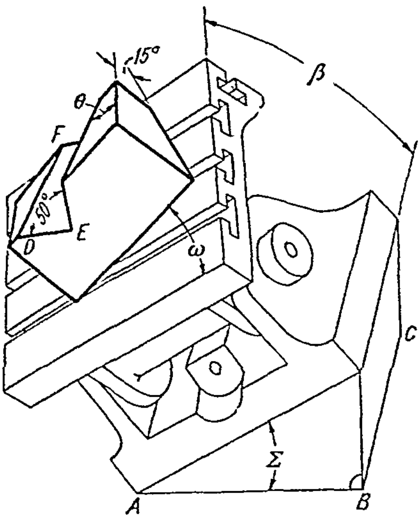


FIG. 57.

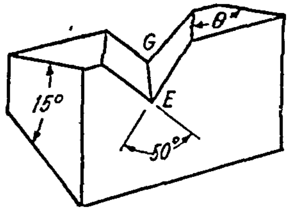


FIG. 58.

VARIABLE

No.	Sym.	Value
1	θ	21°
2	θ	23°
3	θ	25°
4	θ	27°
5	θ	29°
6	θ	31°

$\theta = 33°$

Ans. $\begin{cases} \omega = 33° \ 54' \ 51'' \\ \beta = 12° \ 32' \ 11'' \\ \Sigma = 68° \ 15' \ 37'' \end{cases}$

1. Determine the angle of tilt Σ.
2. Determine the angle of tilt β.
3. Determine the angle of rotation ω.

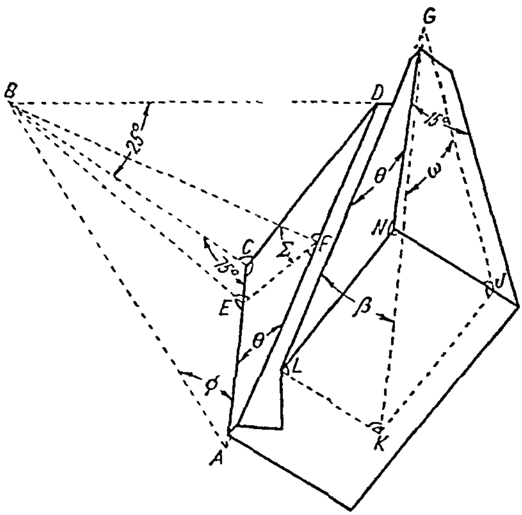


FIG. 59.

GENERAL PROCEDURE FOR COMPUTING ANGLES OF TILT AND ANGLES OF ROTATION

1. Usually a part to be machined must be set up so as to:

a. Have a plane parallel to the base of the fixture.

a′. Have a plane perpendicular to the base of the fixture.

b. Have a line parallel to the base of the fixture.

b′. Have a line perpendicular to the base of the fixture.

Decide, from the requirements of the problem, which of these four conditions is to be fulfilled.

2*a*. For cases *a* and *a*′ set the part on the tilting plane so that the line of intersection of the surface to be machined and the tilting plane (or a plane parallel to the tilting plane) will be parallel to the tilting axis of the fixture.

2*b*. For cases *b* and *b*′ set the part on the tilting plane so that a construction plane containing the line of *b* or *b*′ will be perpendicular to the base of the fixture and to the tilting axis.

3*a*. For cases *a* and *a*′ construct a plane perpendicular to the line of intersection mentioned in 2*a*. The lines of intersection of this plane with the plane to be machined and the tilting plane (or a plane parallel to the tilting plane) form an angle equal to the tilting angle in case *a* and to the complement of the tilting angle in case *a*′.

3*b*. The angle between the line of reference (the line of *b* or *b*′) and the line of intersection of the plane of 2*b* and the tilting plane is the tilting angle for case *b* and is the complement of the tilting angle for case *b*′.

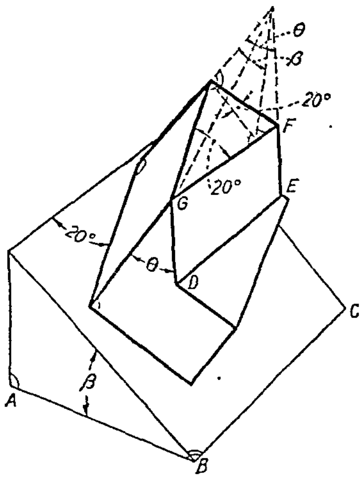
4*a* and 4*b*. The angle of rotation is the angle between an edge of the part in the tilting plane and the tilting axis or a line in the tilting plane perpendicular to the tilting axis.

5*a* and 5*b*. Draw additional construction lines (if necessary) to form one or more of the five fundamental figures which will contain the given angles, an angle equal to the tilting angle, and an angle equal to the angle of rotation, and the surface or line to be machined.

6*a* and 6*b*. Solve for the required angles of tilt and rotation in the usual manner. This may necessitate solving for other angles first.

PROBLEMS

In some of the following problems, the part is shown mounted on fixtures commonly used in tool or die rooms, and in other problems the part is placed on a wedge to show the angles of tilt and rotation.



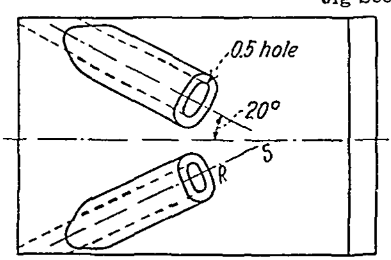
VARIABLE		
No.	Sym.	Value
1	θ	35°
2	θ	37°
3	θ	39°
4	θ	41°
5	θ	43°
6	θ	45°

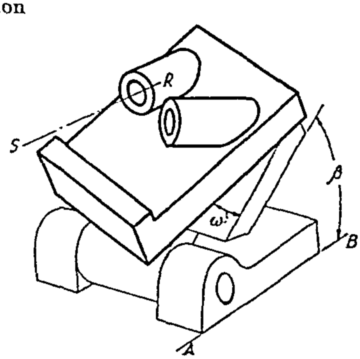
$\theta = 47°$
Ans. $\beta = 45° \; 13' \; 10''$

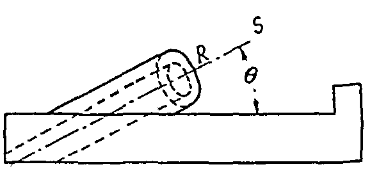
4. Determine the angle β in order that the plane *DEFG* will be perpendicular to the plane *ABC* and parallel to the line *BC*.

Note: The construction lines given indicate the simplicity of this problem.

Jig Section







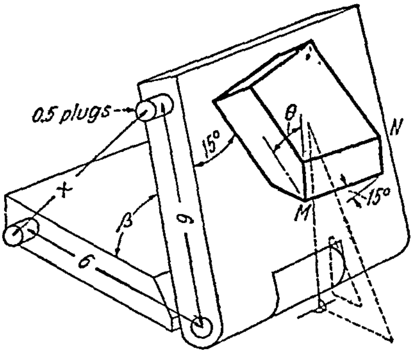
$\theta = 20°$
Ans. $\beta = 18° \; 52' \; 53''$

VARIABLE

1. $\theta = 8°$	2. $\theta = 10°$	3. $\theta = 12°$
4. $\theta = 14°$	5. $\theta = 16°$	6. $\theta = 18°$

The line *RS* of the jig section is to be set parallel with the line *AB* of the base of the adjustable angle plate shown at the right of the jig section.

5. Determine the angle β. 6. Determine the angle ω.



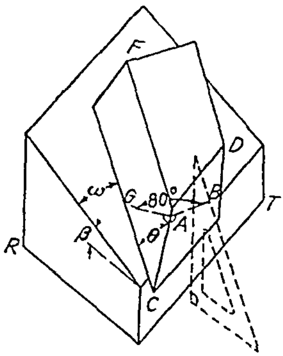
VARIABLE		
No.	Sym.	Value
1	θ	31°
2	θ	33°
3	θ	35°
4	θ	37°
5	θ	39°
6	θ	41°

$$\theta = 29°$$

Ans. $\begin{cases} \beta = 60° \, 8' \, 55'' \\ x = 6.0134 \end{cases}$

Plane MN must be perpendicular to the base.

11. Determine the angle β. **12.** Determine the distance x.



VARIABLE		
No.	Sym.	Value
1	θ	26°
2	6	28°
3	θ	30°
4	θ	32°
5	θ	34°
6	θ	36°

$$\theta = 38°$$

Ans. $\begin{cases} \omega = 15° \, 58' \, 55'' \\ \beta = 50° \, 54' \, 9'' \end{cases}$

The plane CD is to be machined perpendicular to the plane RT. The line AB is in the plane CD and line AG is in the plane CF.

13. Determine the angle of rotation ω. **14.** Determine the angle of tilt β.

ANGULAR PLANES BROUGHT PARALLEL TO BASE OF ANGLE PLATE

If two edges of a surface lying in an angular position are to be brought in a surface parallel to the base of an adjustable angle plate, then a second tilting axis becomes necessary. Assuming, after tilting, that one edge is parallel to the base of the angle plate and the other edge is in, or parallel to, the tilting surface and makes an angle of ω degrees (Fig. 60), the solution for computing the second tilting angle is as follows:

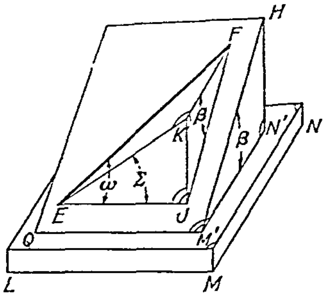
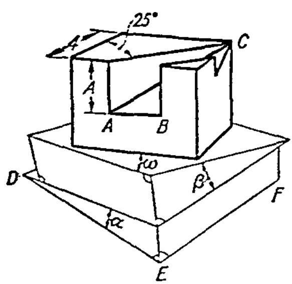


FIG. 60.

Consider the line EF in the tilted surface QH or parallel to it and the line KF in a plane parallel to the base. The plane EFK is to be made parallel to the plane LN. QM' is the first tilting axis and $M'N'$ the second tilting axis. Construct the following lines: EJ parallel to the first tilting axis QM', JF parallel to the line $M'H$, JK parallel to the line $N'H$, and KF parallel to the second tilting axis $M'N'$. Draw the line EK. The tetrahedron EJK-F is a Type I figure with the angles β and ω given. Compute the angle Σ. Tilting the surface QN' about the axis $M'N'$ through an angle of Σ degrees will make the line EF parallel to the horizontal plane.

Verify the following formula:

$$\tan \Sigma = \sin \beta \tan \omega.$$



VARIABLE		
No.	Sym.	Value
1	A	625
2	A	75
3	A	875
4	A	1 000
5	A	1 125
6	A	1.25

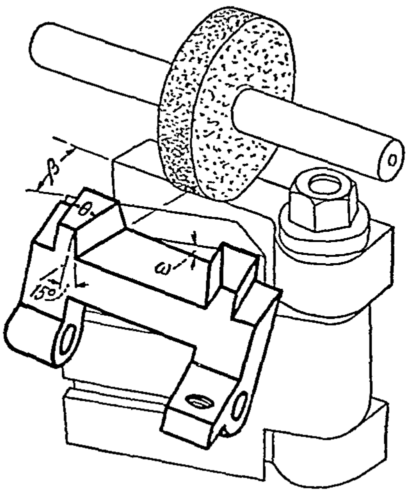
$A = 1.375$

Ans. $\begin{cases} \beta = 17° \, 18' \, 15'' \\ \alpha = 7° \, 53' \, 45'' \end{cases}$

In the above die section the plane ABC must be machined parallel to the plane DEF.

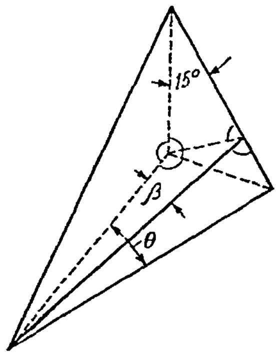
15. Determine the angle β. **16.** Determine the angle α.

17. Determine the angle ω.



VARIABLE

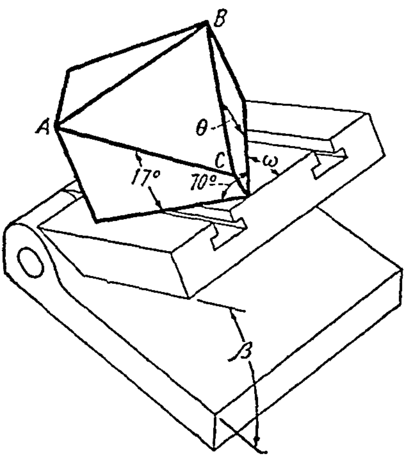
No.	Sym.	Value
1	θ	8°
2	θ	10°
3	θ	12°
4	θ	14°
5	θ	16°
6	θ	18°



Diagrammatic Hint

$\theta = 20°$
Ans. $\beta = 19° \ 22' \ 10''$

18. Determine the angle β.



VARIABLE

No.	Sym.	Value
1	θ	19°
2	θ	20°
3	θ	21°
4	θ	22°
5	θ	23°
6	θ	24°

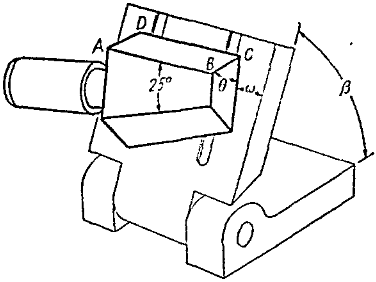
$\theta = 25°$

Ans. $\begin{cases} \beta = 26° \ 10' \ 43'' \\ \omega = 71° \ 32' \ 38'' \end{cases}$

The plane ABC is parallel to the base of adjustable angle plate.

19. Determine the angle β.

20. Determine the angle ω.



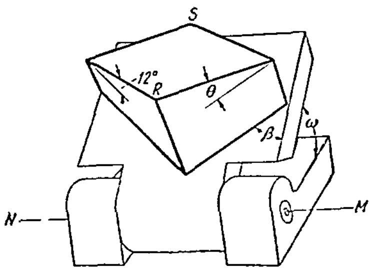
VARIABLE		
No.	Sym.	Value
1	θ	53°
2	θ	55°
3	θ	57°
4	θ	59°
5	θ	61°
6	θ	63°

$\theta = 65°$
Ans. $\beta = 65° \, 31' \, 21''$

21. Determine the angle β in order that the surface $ABCD$ will be parallel to the base of the angle plate.

Derive the following formula:

$$\tan \beta = \tan \theta \sec 12° \, 30'.$$



VARIABLE		
No.	Sym.	Value
1	θ	12°
2	θ	13°
3	θ	14°
4	θ	15°
5	θ	16°
6	θ	17°

$\theta = 18°$

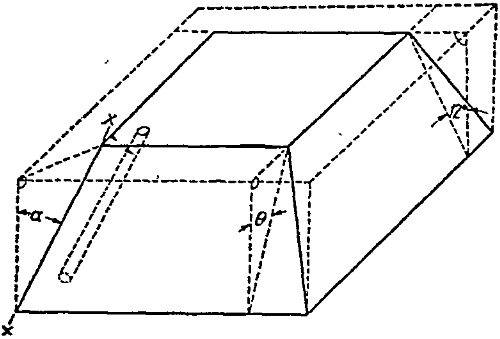
Ans. $\begin{cases} \beta = 33° \, 11' \, 33'' \\ \omega = 21° \, 13' \, 10'' \end{cases}$

Plane RS is parallel to the base of the adjustable angle plate.

22. Determine the angle of rotation β.

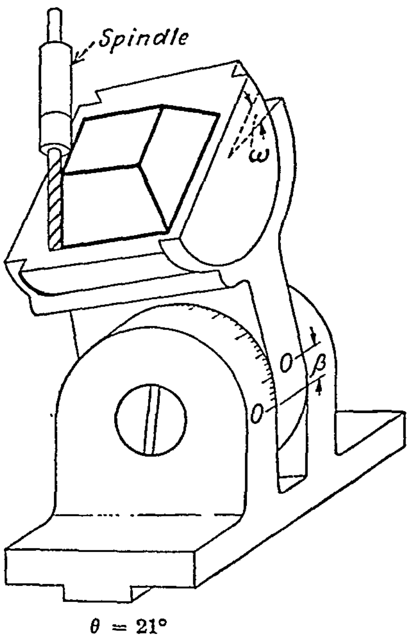
23. Determine the angle of tilt ω.

Die Section



VARIABLE		
No.	Sym.	Value
1	θ	22°
2	θ	23°
3	θ	24°
4	θ	25°
5	θ	26°
6	θ	27°

Mount the die section on the fixture as shown in the following diagram so that a hole may be bored parallel to the xx edge. (The angles on opposite sides of the block are equal.)



$$\theta = 21°$$

$$Ans. \begin{cases} \omega = 21° \text{ or } 12° \\ \beta = 11° \, 13' \, 26'' \text{ or } 20° \, 34' \, 29'' \\ \alpha = 23° \, 41' \, 30'' \end{cases}$$

24. Determine the angle ω.

25. Determine the angle β.

26. Determine the angle α.

Drill Jig Block for Drilling Angular Hole in Bracket

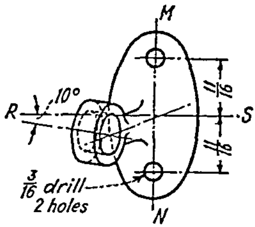


Fig. 61.

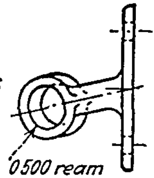


Fig. 62.

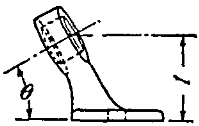


Fig. 63.

Variable		
No.	Sym.	Value
1	θ	24°
2	θ	25°
3	θ	26°
4	θ	27°
5	θ	28°
6	θ	29°

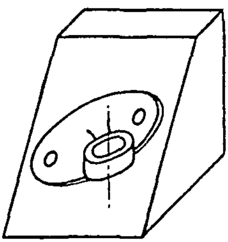


Fig. 64.

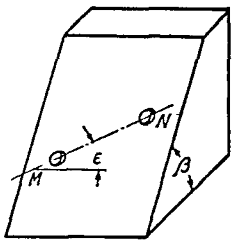


Fig. 65.

$$\theta = 30^\circ$$

$$Ans. \begin{cases} \epsilon = 10^\circ \\ \beta = 60^\circ\ 22'\ 35'' \end{cases}$$

The block of Fig. 64 must be machined so that, after placing bracket on the inclined surface, the .500 hole may be drilled perpendicular to the base of the block.

27. Determine the angle ϵ.

28. Determine the angle β.

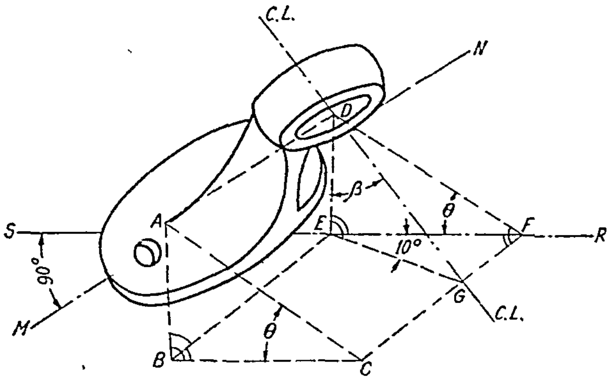


FIG. 66.

Solution: The plane AE is drawn through D perpendicular to the base and parallel to the axis MN. The plane AF is passed through the center line and through AD which is parallel to MN forming the wedge, having right angles as indicated in Fig. 66.

A plane passed through DE and the center line cuts the base in the line EG which forms the angle of 10° with the axis RS.

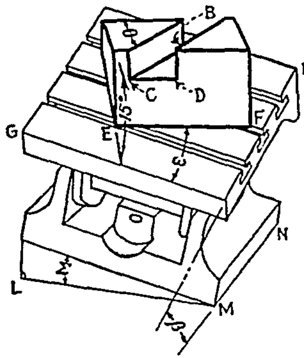
In order properly to mount the bracket on the angular block (Fig. 64), the bracket must be rotated about the axis DE until the line EG coincides with axis RS. This angle of rotation is 10° as given in Fig. 61. This rotation also causes the line through the centers of the two small holes to rotate 10° with respect to the stationary axis MN which is the angle ϵ of Fig. 65.

Since the axis of the drill is perpendicular to the base, the line DG must be made perpendicular to the base; *i.e.*, it must be tilted through the angle β which is equal to the tilting angle of the block of Fig. 65.

The determination of the angle β is accomplished by using the pyramid EGF-D which is of Type I having θ and the 10° angle known.

TOOL-HOLDER BLOCK

In order to machine the tool holder mounted on the adjustable angle plate shown on the top of page 108, the tool holder should be rotated and the angle plate tilted so that the surface BCD, when finished, will be parallel to the base LMN. The sides of the slot are perpendicular to the surface BCD; the line CD is parallel to the base of the tool-holder block.



Variable		
No.	Sym.	Value
1	θ	53°
2	θ	55°
3	θ	57°
4	θ	59°
5	θ	61°
6	θ	63°

$$\theta = 65^\circ$$

$$Ans. \begin{cases} \beta = 13^\circ\ 33'\ 58'' \\ \omega = 25^\circ\ 46'\ 12'' \\ \Sigma = 6^\circ\ 27'\ 38'' \end{cases}$$

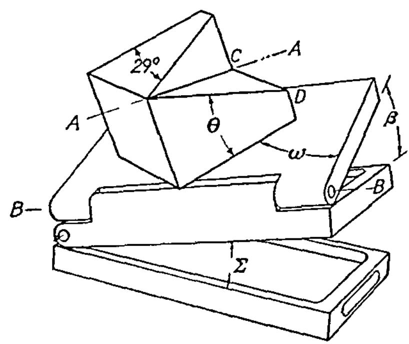
29. Determine the angle β. **30.** Determine the angle ω.

31. Determine the angle Σ.

Verify the following formulas:

$$\sin \beta = \sin 15^\circ \sin \theta, \qquad \tan \omega = \cot \theta \sec 15^\circ,$$

$$\tan \Sigma = \sin \beta \tan \omega.$$ (See Fig. *a*, page 66.)



$$\theta = 27^\circ$$

$$Ans. \begin{cases} \omega = 29^\circ \\ \beta = 24^\circ\ 1'\ 9'' \\ \Sigma = 12^\circ\ 42'\ 52'' \end{cases}$$

Variable

1. $\theta = 15^\circ$	**2.** $\theta = 17^\circ$	**3.** $\theta = 19^\circ$
4. $\theta = 21^\circ$	**5.** $\theta = 23^\circ$	**6.** $\theta = 25^\circ$

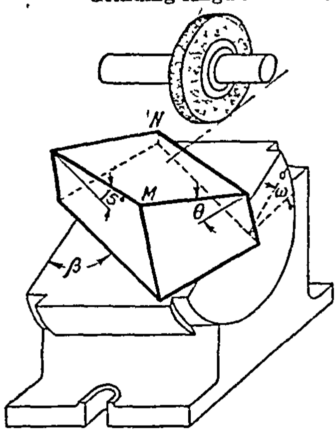
The line CD is parallel to the base of the die section.

32. Determine the angle ω.

33. Determine the angle β.

34. Determine the angle Σ. (See Fig. 60, page 102.)

Grinding Angular Surface on Block



Variable		
No.	Sym.	Value
1	θ	22°
2	θ	23°
3	θ	24°
4	θ	25°
5	θ	26°
6	θ	27°

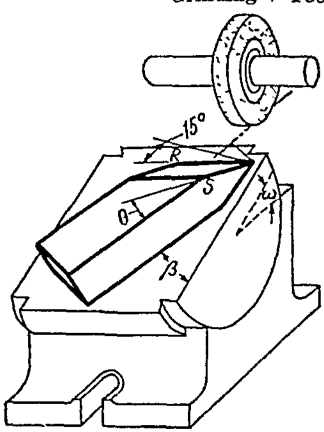
$\theta = 21°$

Ans. $\begin{cases} \beta = 55° \, 4' \, 54'' \\ \omega = 25° \, 5' \, 12'' \end{cases}$

Plane *MN* is parallel to the base of fixture.

35. Determine the angle β.
36. Determine the angle ω.

Grinding V Tools



Variable		
No.	Sym.	Value
1	θ	29°
2	θ	30°
3	θ	31°
4	θ	32°
5	θ	33°
6	θ	34°

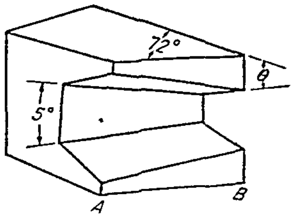
$\theta = 28°$

Ans. $\begin{cases} \beta = 15° \\ \omega = 28° \, 49' \, 53'' \end{cases}$

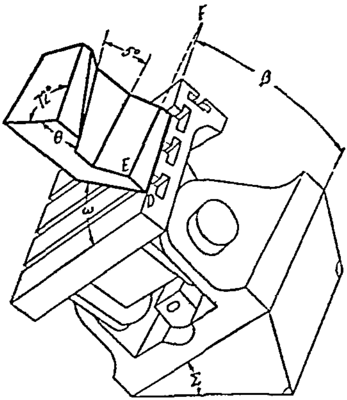
Plane *RS* is parallel to the base of fixture.

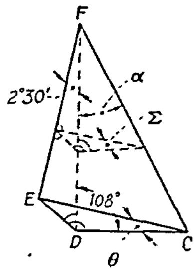
37. Determine the angle β.
38. Determine the angle ω.

Tool for Backing Off Teeth on Hobs



Variable		
No.	Sym.	Value
1	θ	11°
2	θ	13°
3	θ	15°
4	θ	17°
5	θ	19°
6	θ	21°





Diagrammatic Hint

$\theta = 23°$

Ans. $\Sigma = 24° \ 42' \ 25''$

The edge AB of the part shown in the upper figure is set parallel to the edge of the fixture.

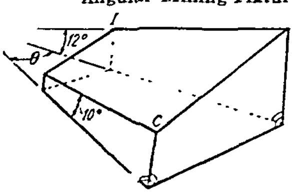
39. Determine the angle Σ.

Hint: Use Types IV and I. Note that $\omega = 90° - 72° = 18°$

$$\text{and } \beta = \frac{5°}{2} = 2° \ 30'$$

Verify: $\tan \Sigma = \cos 2° \ 30' \tan \theta \csc 72° + \cot 72° \sin 2° \ 30'$.

Angular Milling Fixture Support



Variable		
No.	Sym.	Value
1	θ	20°
2	θ	22°
3	θ	24°
4	θ	26°
5	θ	28°
6	θ	30°

Fig. 67.

40. Determine the angle of rotation ω as shown in Fig. 70.

(*This problem continued on next page.*)

41. Determine the angle of tilt α (of Fig. 70) in order to grind the surface CI (of Fig. 67) parallel to the base of the adjustable angle plate shown in Fig. 70.

Solution to Problems 40 *and* 41:

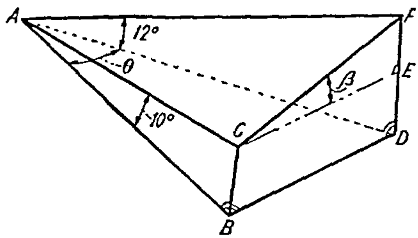


FIG. 68.

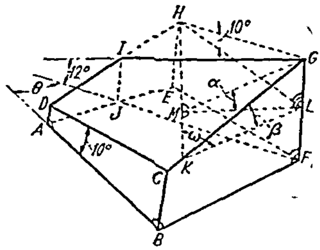


FIG. 69.

From Fig. 68 show that: $\tan \beta = \tan 12° \csc \theta - \cot \theta \tan 10°$.

The plane FH is constructed parallel to the plane BD, and therefore the angle GHL is equal to 10°. The figure $HKLG$ is of Type II.

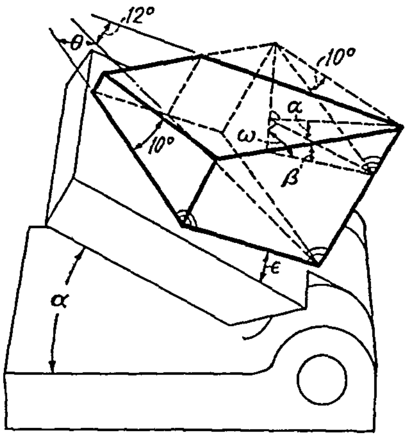
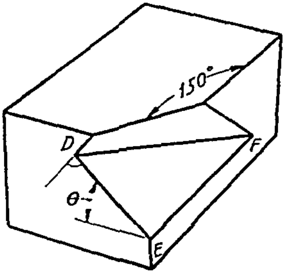


FIG. 70.

$$\theta = 32°$$

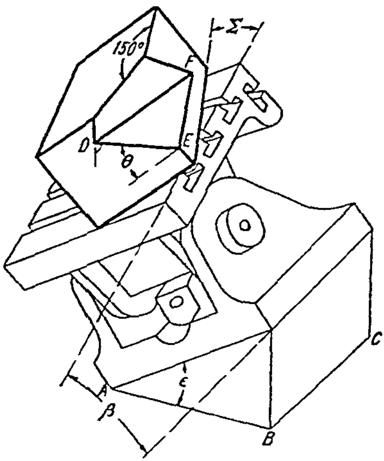
$$Ans. \begin{cases} \omega = 56° \; 0' \; 15'' \\ \alpha = 12° \; 0' \; 34'' \end{cases}$$

$$Verify: \begin{cases} \tan \omega = \cot 10° \tan 12° \csc \theta - \cot \theta. \\ \tan \alpha = \csc \omega \, (\tan 12° \csc \theta - \cot \theta \tan 10°). \end{cases}$$



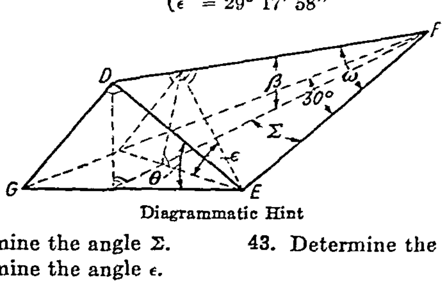
VARIABLE		
No.	Sym.	Value
1	θ	20°
2	θ	22°
3	θ	24°
4	θ	26°
5	θ	28°
6	θ	30°

The die section shown above is to be placed on the double tilting angle plate in such a manner that the plane *EFD* will be parallel to the base *ABC* of the adjustable angle plate.



$\theta = 32°$

Ans. $\begin{cases} \Sigma = 22° \; 32' \; 56'' \\ \beta = 13° \; 28' \; 30'' \\ \epsilon = 29° \; 17' \; 58'' \end{cases}$



Diagrammatic Hint

42. Determine the angle Σ.

43. Determine the angle β.

44. Determine the angle ϵ.

In connection with the preceding problem,

Verify: $\begin{cases} \tan \omega = \cos \theta \tan 30°. & \tan \Sigma = \cos^2 \theta \tan 30°. \\ \sin \beta = \sin \theta \sin \omega. & \sin \epsilon = \tan \beta \sec \theta \cot 30°. \end{cases}$

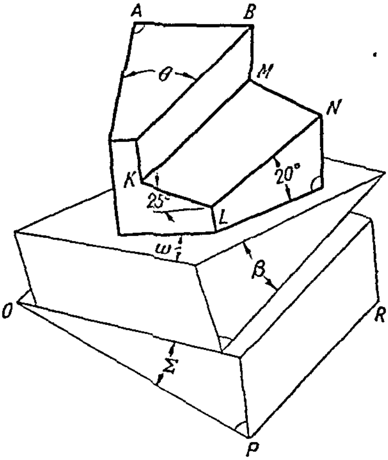


FIG. 71.

Variable		
No.	Sym.	Value
1	θ	22°
2	θ	24°
3	θ	26°
4	θ	28°
5	θ	30°
6	θ	32°

$\theta = 34°$

Ans. $\begin{cases} \beta = 20° \\ \omega = 36° \; 31' \; 5'' \\ \Sigma = 38° \; 36' \; 22'' \end{cases}$

The die section must be set up on the double tilting device in such a manner that the plane KN is parallel to the base (RO) and the line KM is parallel to the edge PR.

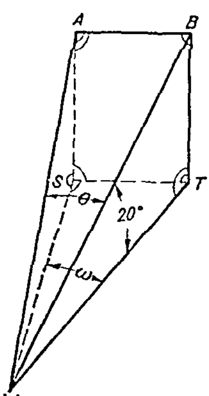


FIG. 72.

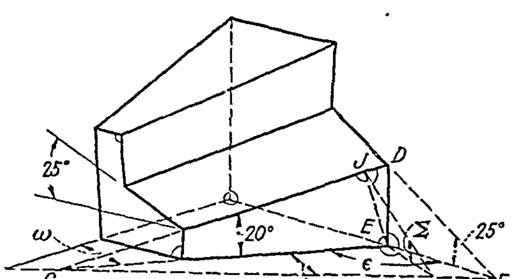


FIG. 73.

45. Determine the angle β.

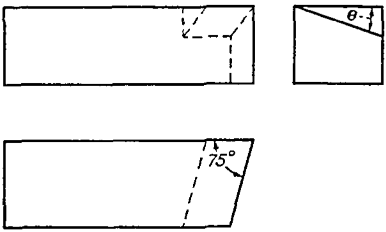
46. Determine the angle ω.

47. Determine the angle Σ.

Hint: Figure 72 represents the left-hand part of the die section. The top surface is extended to meet a plane *STW* which is parallel to the base of the die section. Solve for the angle of rotation ω.

In Fig. 73 the angle ϵ is seen to be $90° + \omega$. Thus the pyramid *CEF-D* is of Type IV having the given angles, ϵ, 20°, and 25°. Solve for the angle Ψ.

Then pass a plane perpendicular to the plane *CDE* and perpendicular to the line *CD* at *J*. Solve the Type I pyramid *CEH–J* for the angle Σ.



VARIABLE		
No.	Sym.	Value
1	θ	8°
2	θ	10°
3	θ	12°
4	θ	14°
5	θ	16°
6	θ	18°

FIG. 74.

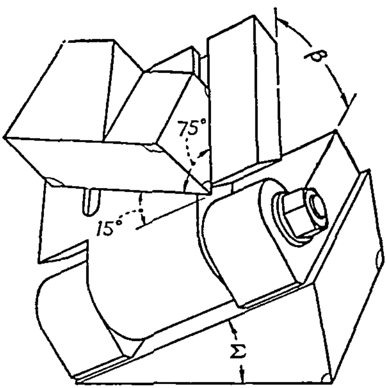


FIG. 75.

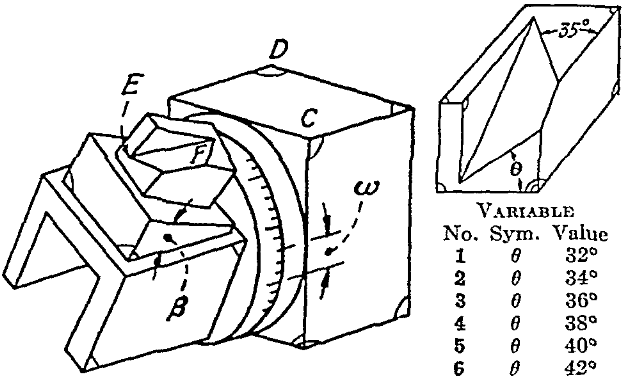
$$\theta = 20°$$

$$Ans. \begin{cases} \beta = 19° \, 22' \, 10'' \\ \Sigma = 5° \, 4' \, 41'' \end{cases}$$

From the data given in the mechanical drawing (Fig. 74):

48. Determine the angle of tilt β.

49. Determine the angle of tilt Σ. (See Fig. 60, page 102.)



VARIABLE

No.	Sym.	Value
1	θ	32°
2	θ	34°
3	θ	36°
4	θ	38°
5	θ	40°
6	θ	42°

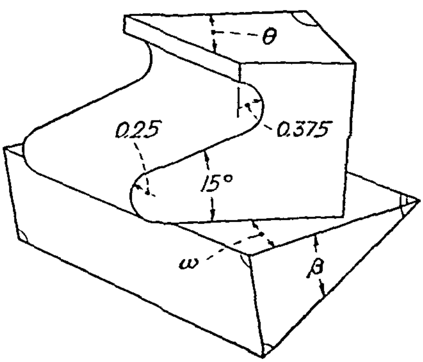
$$\theta = 44^\circ$$

$$Ans. \begin{cases} \beta = 34^\circ\ 3'\ 57'' \\ \omega = 38^\circ\ 39'\ 27'' \end{cases}$$

Plane *EF* of the block to be shaped is parallel to the plane *CD* which is parallel to the plane of motion of the ram of the shaper.

50. Determine the angle β.

51. Determine the angle ω.



VARIABLE

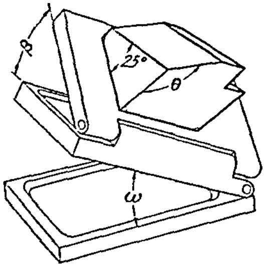
No.	Sym.	Value
1	θ	65°
2	θ	67°
3	θ	69°
4	θ	71°
5	θ	73°
6	θ	75°

$$\theta = 77^\circ$$

$$Ans. \begin{cases} \omega = 13^\circ \\ \beta = 15^\circ\ 22'\ 21'' \end{cases}$$

52. Determine the angle ω.

53. Determine the angle β.



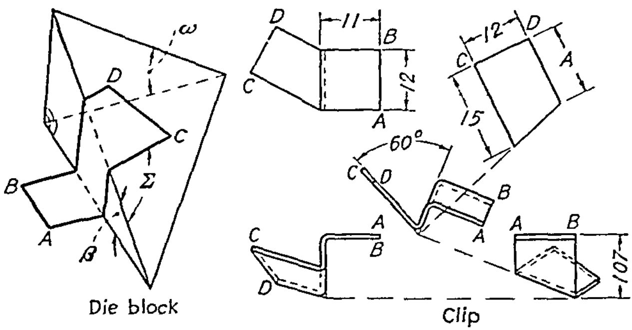
VARIABLE

No.	Sym.	Value
1	θ	102°
2	θ	103°
3	θ	104°
4	θ	105°
5	θ	106°
6	θ	107°

$$Ans. \begin{cases} \theta = 108^\circ \\ \beta = 25^\circ \\ \omega = 16^\circ\ 24'\ 27'' \end{cases}$$

54. Determine the angle ω.

Luggage-compartment Clip



$$Ans \begin{cases} A = 1\ 250 \\ \beta = 11^\circ\ 18'\ 36'' \\ \Sigma = 101^\circ\ 18'\ 36'' \\ \omega = 30^\circ\ 29'\ 20'' \end{cases}$$

VARIABLE

1. $A = 1.034$	2. $A = 1\ 070$	3. $A = 1.106$
4. $A = 1.142$	5. $A = 1.178$	6. $A = 1.214$

The angles β, Σ, and ω are necessary in order to make the die block which in turn forms the clip.

55. Determine the angle β.

56. Determine the angle Σ.

57. Determine the angle ω.

Shaping and Grinding a Die Section

Owing to the construction of the different machines and the available equipment, the set-up angles are likely to be different for the machining of the same or parallel surfaces on different types of machinery, such as shaper, grinder, etc.

The following problem will serve as an illustration. Consider the sectional block below to be machined first on a shaper and later finished on a grinder.

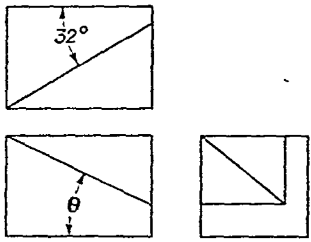
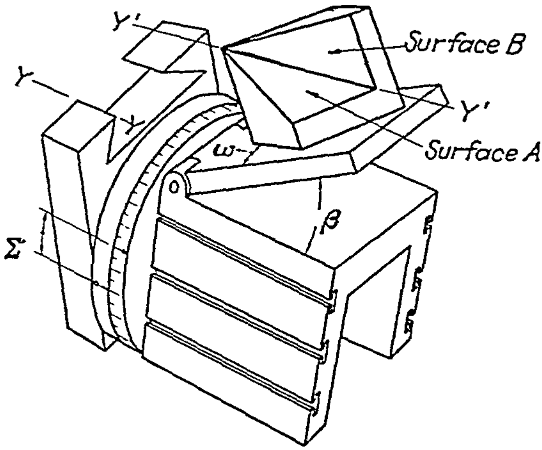


FIG. 76.

The figure below shows the necessary tilting angle β and rotation angles ω and Σ for machining the surface A of the sectional block on a shaper. These tilting and rotation angles are necessary in order that the lines YY and $Y'Y'$ and the surface A be parallel to the path of the tool and the cross feed of the shaper, respectively.



VARIABLE

No.	Sym.	Value
1	θ	22°
2	θ	23°
3	θ	24°
4	θ	25°
5	θ	26°
6	θ	27°

$\theta = 21°$

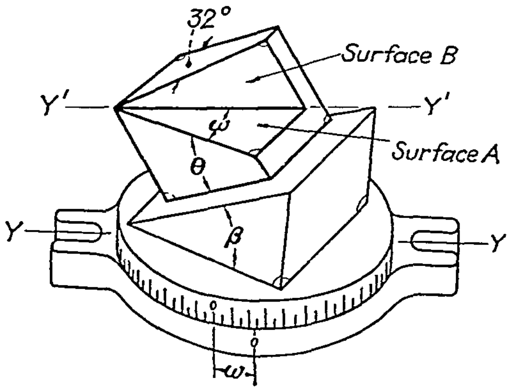
$$Ans. \begin{cases} \beta = 18° \ 1' \ 54'' \\ \omega = 32° \\ \Sigma = 10° \ 56' \ 47'' \end{cases}$$

58. Determine the angle of tilt β.
59. Determine the first angle of rotation ω.
60. Determine the second angle of rotation Σ.

Now consider the machining of surface B in the preceding figure, the feed moving vertically without changing the position of the sectional block.

61. Determine the angle of rotation Σ of the foregoing figure.

In order to grind the surface A of the sectional block of the preceding problem, the lines YY and $Y'Y'$ must be parallel to the path of the grinding wheel. Hence the fixture must be tilted $\beta°$ and rotated $\omega°$.



$$\theta = 21°$$

Ans. $\begin{cases} \beta = 21° \\ \omega = 30° \; 15' \; 29'' \end{cases}$

VARIABLE

1. $\theta = 22°$	**2.** $\theta = 23°$	**3.** $\theta = 24°$
4. $\theta = 25°$	**5.** $\theta = 26°$	**6.** $\theta = 27°$

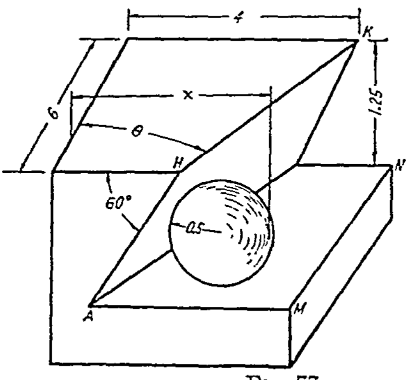
62. Determine the angle of tilt β.
63. Determine the angle of rotation ω.

CHAPTER III

MISCELLANEOUS APPLICATIONS OF COMPOUND ANGLES

In addition to the many practical applications of compound angles involved in the problems of Chaps. I and II, there are several of particular importance which require more explanation than was given the problems of the previous chapters. In this chapter, such problems as checking angular tapered plugs by balls, serrated tapered gages, angles of a positive clutch, side mill cutters, facing cutters, making of radial flat tools, making of flat form tools, and miscellaneous problems solved by the projection method are discussed.

CHECKING ANGULAR TAPERED DOVETAILS BY MEANS OF BALLS



Variable		
No.	Sym.	Value
1	θ	15°
2	θ	17°
3	θ	19°
4	θ	21°
5	θ	23°
6	θ	25°

Fig. 77.

The 1-in. ball is tangent to the two planes AHK and AMN, and flush with the plane HAM.

The angular tapered dovetail cannot be checked by direct measurement, but the distance x can be measured and hence the value of x must be computed.

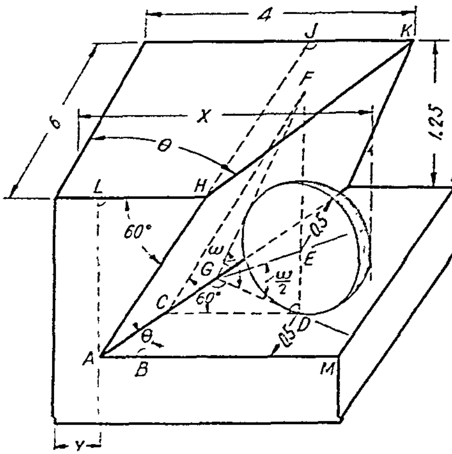


FIG. 78.

$\theta = 27°$

Ans. $x = 1.8956$

The plane FGD is passed through the center of the sphere perpendicular to the edge AG.

$y = 4 - JK - HL = 4 - 6 \tan \theta - 1.25 \cot 60°.$

$x = y + AB + CD + .5.$

$\tan \omega = \tan 60° \sec \theta. \qquad GD = .5 \cot \frac{\omega}{2}. \qquad CD = GD \sec \theta.$

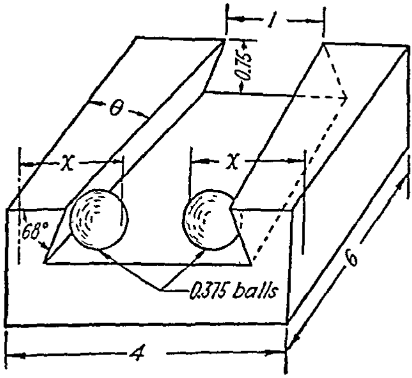
$AB = .5 \tan \theta.$

$$x = y + .5 \tan \theta + .5 \cot \frac{\omega}{2} \sec \theta + .5.$$

The student should thoroughly check each step of the foregoing solution.

PROBLEMS

1. Determine x in the foregoing illustrative problem.

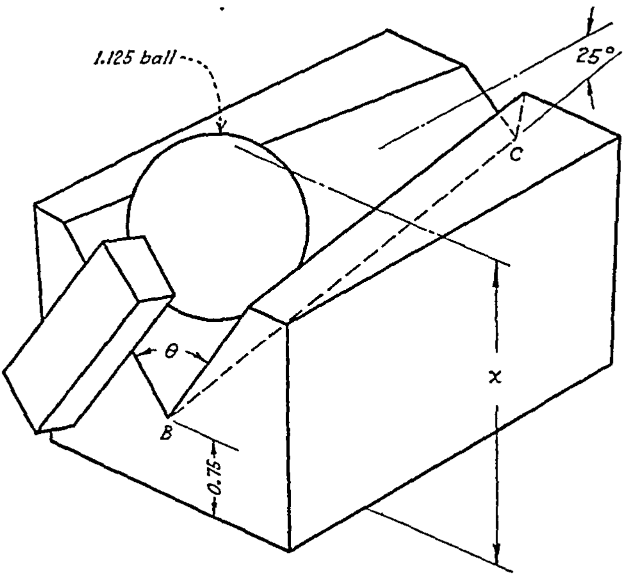


Variable		
No.	Sym.	Value
1	θ	5°
2	θ	6°
3	θ	7°
4	θ	8°
5	θ	9°
6	θ	10°

$\theta = 11°$
Ans. $x = .53588$

The balls are flush with the front face of the tapered dovetail slide.

2. Determine the distance x.



$\theta = 84°$
Ans. $x = 2.4554$

Variables

1. $\theta = 72°$	2. $\theta = 74°$	3. $\theta = 76°$
4. $\theta = 78°$	5. $\theta = 80°$	6. $\theta = 82°$

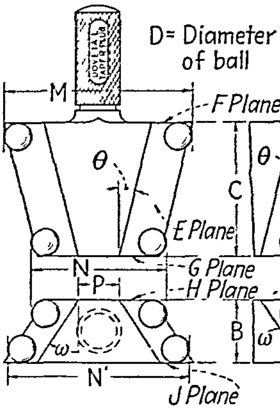
3. Determine the distance x.

Hint: Determine the angle perpendicular to the line BC. (*See formulas at top of next page.*)

$$Verify\begin{cases}\tan\beta = \tan\dfrac{\theta}{2}\sec 25^\circ \\ x = R(\csc\beta\sec 25^\circ + \tan 25^\circ + 1) + .75\end{cases}$$

where β is one-half the true angle between the planes tangent to the ball and R the radius of the ball.

CHECKING ANGULAR TAPERED PLUG GAGES BY MEANS OF BALLS



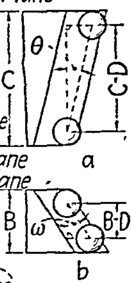


FIG. 79. FIG. 80.

For the value of N the balls are tangent to the E plane and flush with the H and G planes. In all cases first compute the value of N.

$$N = D\left[\cot\frac{\phi}{2}\sec\theta + \tan\theta + 1\right] + P$$

Where $\tan\phi = \cot\omega\sec\theta$

For the value of N' the balls are tangent to the E plane and flush with the G and J planes.

$$N' = N + 2(B - D)\tan\omega \qquad \text{(See Fig. 80}b)$$

For the values of M the balls are tangent to the E plane and sh with the F and H planes.

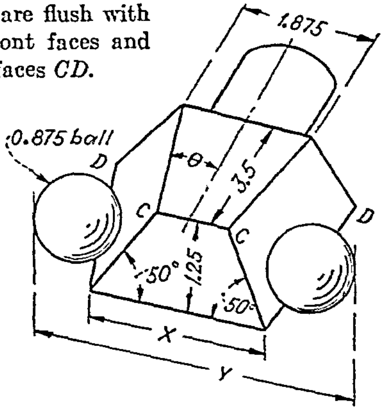
$$M = N + 2(C - D)\tan\theta \qquad \text{(See Fig. 80}a)$$

For the value of M' the balls are tangent to the E plane and sh with the F and J planes.

$$M' = M + 2(B - D)\tan\omega = N' + 2(C - D)\tan\theta$$

PROBLEMS

The .875 balls are flush with the lower and front faces and tangent with the faces *CD*.



VARIABLE		
No.	Sym.	Value
1	θ	2° 30′
2	θ	3° 30′
3	θ	4° 30′
4	θ	5° 30′
5	θ	6° 30′
6	θ	7° 30′

$\theta = 8° 30'$

Ans. $\begin{cases} x = 2.9266 \\ y = 4.3477 \end{cases}$

4. Determine the distance x.

5. Determine the distance y.

CHECKING SERRATED TAPERED GAGES BY MEANS OF BALLS

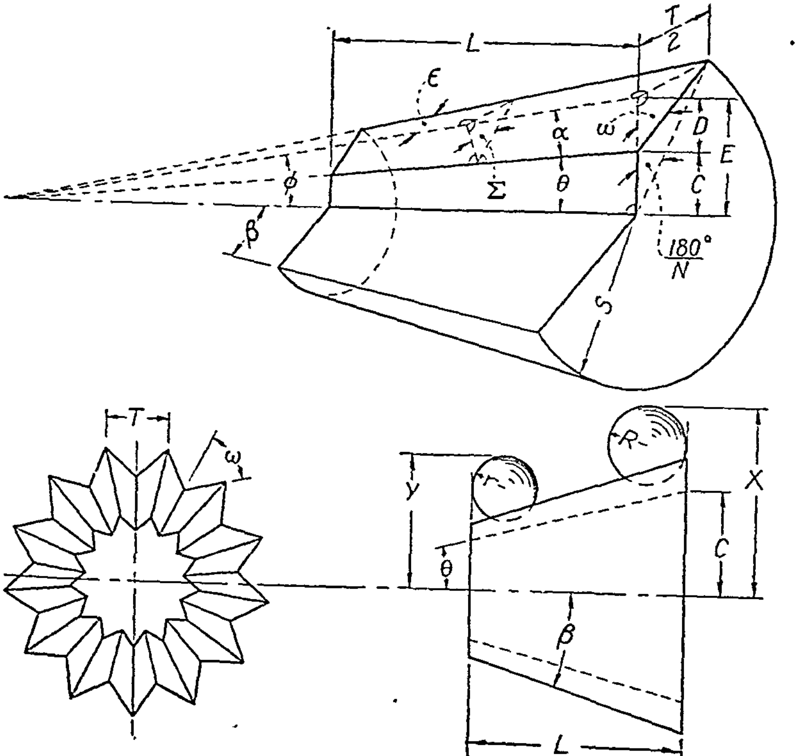


FIG. 81.

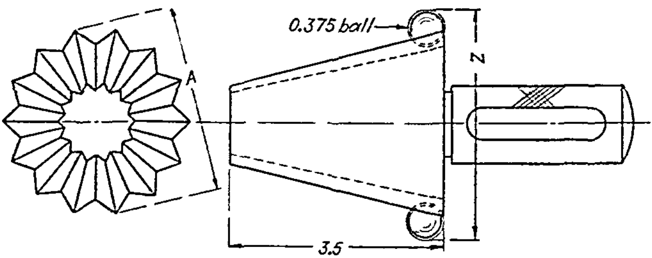
GIVEN PARTS

Notation			
N	Number of serrations	L	Length of gage
T	Chordal width at large end of tooth	Σ	$\frac{1}{2}$ angle of cutter to be used
R and r	Radii of balls	β	Angle from center line to outer surface

Formulas for computing the distances x and y:

1. $\sin \epsilon = \sin \beta \sin \dfrac{180^\circ}{N}$.
2. $\sin \alpha = \cot \Sigma \tan \epsilon$.
3. $\tan \phi = \tan \beta \cos \dfrac{180^\circ}{N}$.
4. $\theta = \phi - \alpha$.
5. $\tan \omega = \tan \Sigma \cos \theta$.
6. $\dfrac{T}{2} = S \sin \dfrac{180^\circ}{N}$.
7. $D = \dfrac{T}{2} \cot \omega$.
8. $E = \dfrac{T}{2} \cot \dfrac{180^\circ}{N}$.
9. $C = E - D$.
10. $x = C + R(\csc \Sigma \sec \theta - \tan \theta + 1)$.
11. $y = C + r(\csc \Sigma \sec \theta + 1) - (L - r) \tan \theta$.

PROBLEM



$A = 3.5$
Ans. $Z = 3.4586$

VARIABLE

1. $A = 2.75$	2. $A = 2.875$	3. $A = 3.00$
4. $A = 3.125$	5. $A = 3.25$	6. $A = 3.375$

$\beta = 15^\circ$ $\Sigma = 45^\circ$ (See Fig. 81)

6. Determine the distance Z.

TILTING AND FACING ANGLES FOR THE MILLING OF POSITIVE CLUTCHES

There are three forms of positive clutches which are as shown in the figure below:

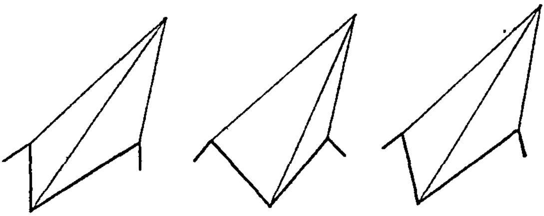


FIG. 82.

The solution for these three forms will be discussed in the order of their complexity.

Form 1

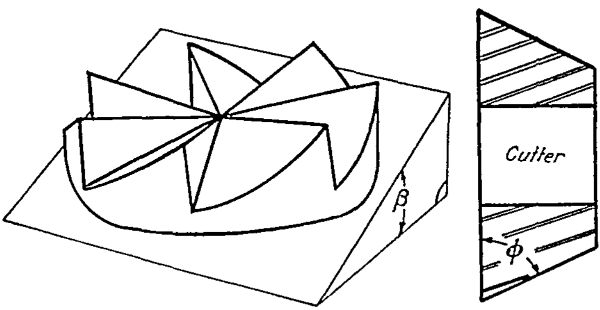


FIG. 83.

In Form 1, one face of a tooth is parallel to the axis of the clutch.

In Fig. 83, which shows the clutch and the cross-sectional view of the cutter, angle β is the tilting angle of the dividing head measured from the vertical position of the dividing head, and angle ϕ is the angle of the cutter.

Angle β is also the angle at which the compound on the lathe must be set in order to face the top of the positive clutch.

N is the number of teeth in the clutch.

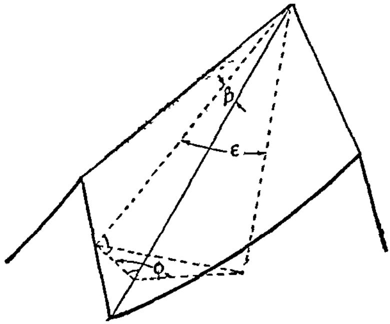


FIG. 84.

The student should show from the enlarged view of a tooth (Fig. 84) that

$$\sin \beta = \cot \phi \tan \varepsilon,$$

where $\epsilon = \frac{180°}{N}$.

PROBLEM

VARIABLE

1. $N = 11$	2. $N = 13$	3. $N = 15$
4. $N = 17$	5. $N = 19$	6. $N = 21$

In a positive clutch (Form 1), the number of teeth is N and the cutter angle is 70°.

7. Determine the tilting angle β (which is also the facing angle).

Form 2

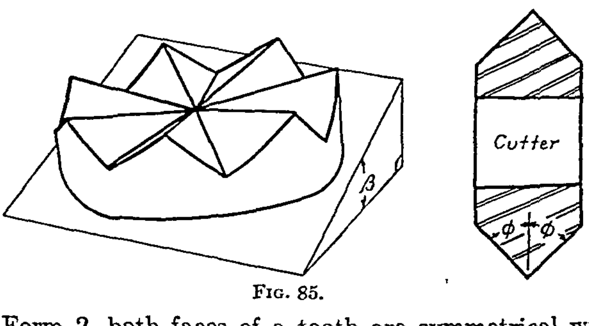


FIG. 85.

In Form 2, both faces of a tooth are symmetrical with a plane passing through the axis of the clutch and the center of a tooth.

$$= \frac{1}{2}[13.2340 - 13.0390] = .0975$$
$$= 5^\circ 35' 45''.$$

PROBLEM

VARIABLE

1. $N = 8$	2. $N = 10$	3. $N = 12$
4. $N = 14$	5. $N = 16$	6. $N = 18$

In a positive clutch (Form 3), the number of teeth is N, $\theta = 30^\circ$ and $\phi = 60^\circ$.

9. Determine the tilting angle β (which is also the facing angle).

Side Mill Cutters

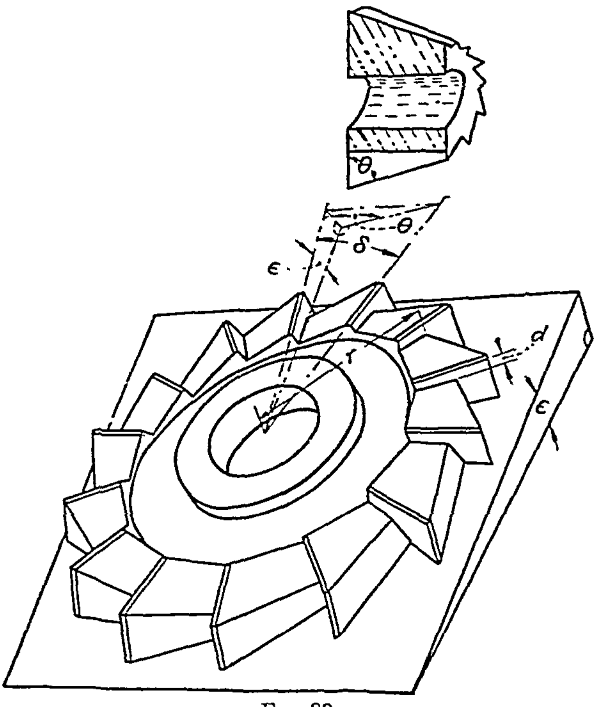


FIG. 89.

In order to machine a side mill cutter, it is necessary to determine the angle of tilt ϵ and the angle of the angular cutter θ, the latter being limited to angles varying by 5°.

NOTATION

N = number of teeth in side mill cutter.
θ = desired angle of angular cutter.
θ' = nearest available angle of angular cutter (which is the one used).
ϵ = desired angle of tilt of side mill cutter.
ϵ' = actual angle of tilt.

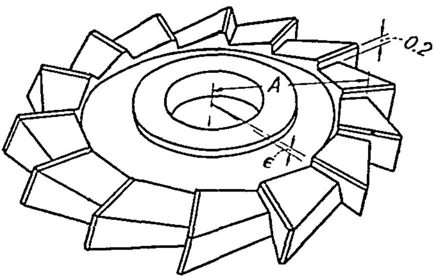
FORMULAS

$$\tan \epsilon = \frac{d}{r}. \qquad \delta = \frac{360°}{N}.$$

$$\tan \theta = \tan \delta \csc \epsilon.$$

$$\sin \epsilon' = \cot \theta' \tan \delta.$$

PROBLEMS



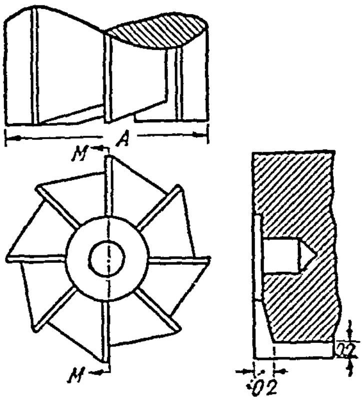
Variable		
No.	Sym.	Value
1	A	2 625
2	A	2.75
3	A	2.875
4	A	3.000
5	A	3.125
6	A	3.25

$A = 2.5$

Ans. $\begin{cases} \theta' = 80° \\ \epsilon' = 4° \, 52' \, 14'' \end{cases}$

10. Determine the angle of cutter θ' used to mill the teeth in side mill cutter, the available angles of cutter varying by 5°.

11. Determine the tilting angle ϵ' used to machine this cutter.



VARIABLE		
No.	Sym.	Value
1	A	2.25
2	A	2.125
3	A	2.000
4	A	1.875
5	A	1.75
6	A	1.625

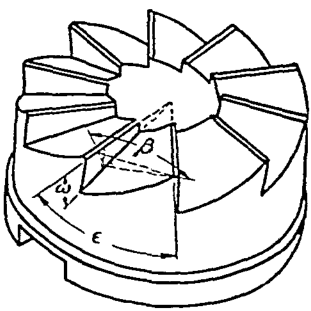
$A = 2.375$

Ans. $\begin{cases} \theta' = 80° \\ \epsilon' = 10° \, 9' \, 32'' \end{cases}$

12. Determine the angle θ' of the cutter used to mill the teeth in end mill, the available angles of cutter varying by 5°.

13. Determine the tilting angle ϵ'.

Facing Cutter



VARIABLE		
No.	Sym.	Value
1	A	3.25
2	A	3.375
3	A	3.5
4	A	3.625
5	A	3.75
6	A	3.875

FIG. 90.

$A = 4$

Ans. $\begin{cases} \epsilon = 36° \\ \omega = 18° \, 29' \, 2'' \\ \beta = 66° \, 25' \, 30'' \end{cases}$

14. Determine the angle ϵ.

15. Determine the angle ω.

16. Determine the angle β.

Solution:

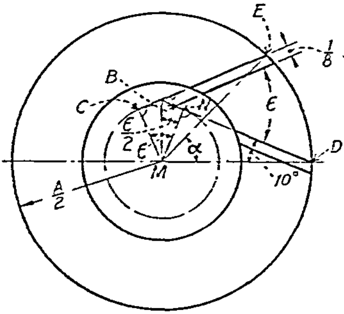


FIG. 91.

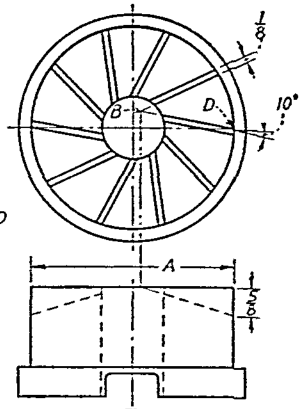


FIG. 92.

In Fig. 91, the lines DB and EC represent the corresponding edges of two consecutive teeth. The small circle is constructed tangent to these two lines. N = number of teeth in cutter.

$$\text{Angle } \alpha = \frac{360°}{N}.$$

But angle α = angle ϵ (corresponding angles of similar triangles).

$$\therefore \text{ angle } \epsilon = \frac{360°}{N}.$$

$$MN = \frac{A}{2} \sin 10°.$$

$$BD = \frac{A}{2}\left(\sin 10° \tan \frac{\epsilon}{2} + \cos 10°\right) - .125 \csc \epsilon$$

β is the angle of the angular cutter which is used to cut the teeth of the facing cutter.

Verify: $\cot \beta = \sin \omega \cot \epsilon$.

MAKING OF RADIAL FLAT TOOLS

Arc of Circle Central with Center Line of Tool

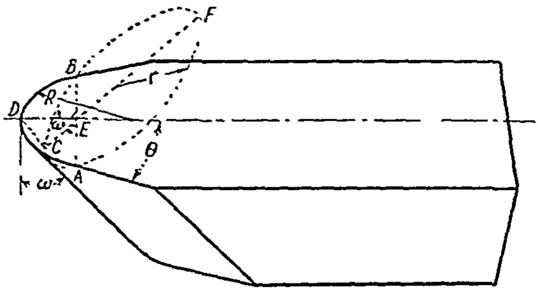


Fig. 93.

The radial flat tool is to have a cutting edge which is a portion of a circle having a radius R, and an angle 2θ between the tangents drawn to the extremities of the cutting arcs, and a tool clearance angle ω. In order to make the tool, the chord AB, which is perpendicular to the center line of tool, and the radius r of the approximate circle formed by passing a plane through this chord perpendicular to the line which forms the clearance angle must be determined. The latter section is really a portion of an ellipse but may be considered with very little error as a portion of a circle if the depth DE of the chord AB is less than half the radius R.

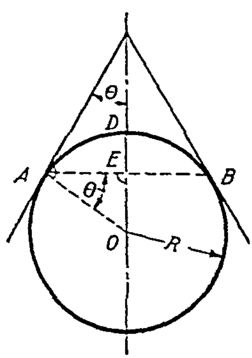


Fig. 94.

In the foregoing figure (Fig. 94) the half chord AE is seen to be $R \cos \theta$ (angle $OAE = \theta$). $DE = R - R \sin \theta$.

In Fig. 93, $EC = (DE) \cos \omega$.

By Proposition 45,

$$EF = \frac{(\frac{1}{2}AB)^2}{EC}.$$

Hence

$$r = \frac{1}{2}\left[\frac{1}{4}\frac{(AB)^2}{CE} + CE\right].$$

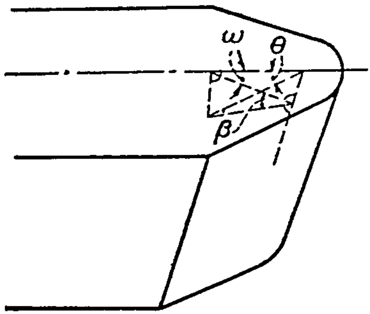
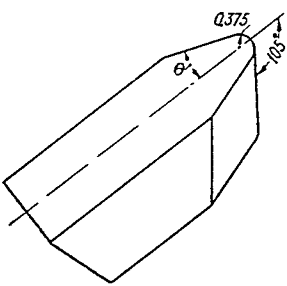


FIG. 95.

In order to make this tool, it is also necessary to determine the compound angle β of Fig. 95 where 2β is the angle between the two angular faces of the tool. Construction lines necessary in determining β are given in Fig 95.

PROBLEMS



VARIABLE

No.	Sym.	Value
1	θ	32°
2	θ	34°
3	θ	36°
4	θ	38°
5	θ	40°
6	θ	42°

$\theta = 44°$

Ans. $\begin{cases} r = .38425 \\ \beta = 44° \, 59' \, 36'' \end{cases}$

17. Determine the radius r of the approximate circle perpendicular to the clearance line.

18. Determine the angle β for the above tool as illustrated in Fig. 95 of the foregoing discussion.

Radial Tool Used on Screw Machine

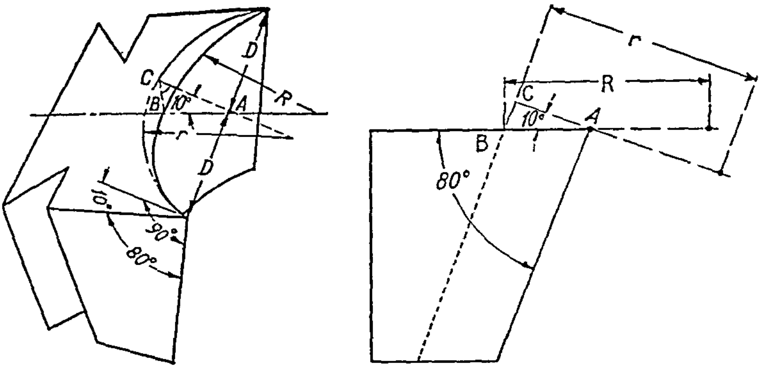
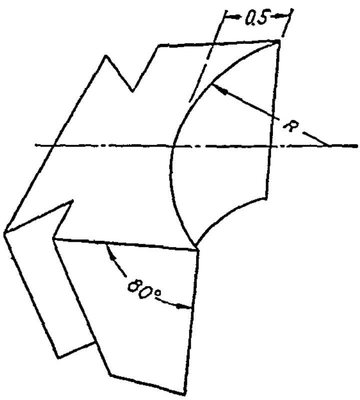


FIG. 96.

In case the radial flat tool has a concave cutting edge, the solution is similar to the one given for the case of a convex cutting edge. The plane passed through the chord perpendicular to the clearance edge is tilted up in this case. The radius of the approximate circle formed in this plane is represented by r in Fig. 96.

PROBLEM



VARIABLE

1.	$R = 2$
2.	$R = 2.125$
3.	$R = 2.25$
4.	$R = 2.375$
5.	$R = 2.5$
6.	$R = 2.625$

$R = 2.75$
Ans. $r = 2.7847$

19. Determine the radius r of the circle whose plane is perpendicular to the clearance edge.

Arc of Circle Not Central with Center Line of Tool

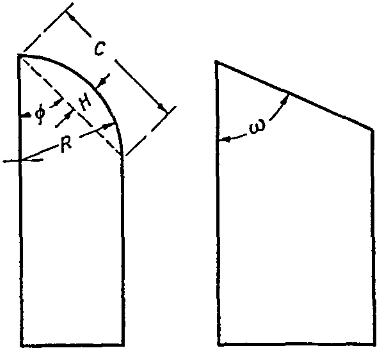


FIG. 97.

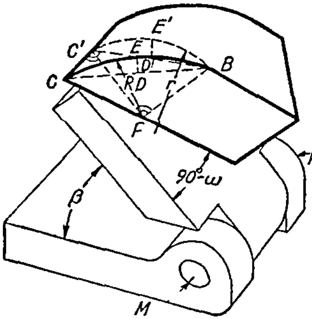


FIG. 98.

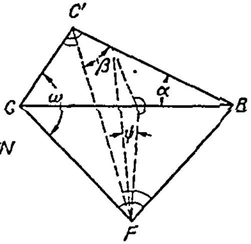


FIG. 99.

In case the chord c is not perpendicular to the center line of the tool but makes an angle ϕ with it, as in Fig. 97, the solution for the radius r (Fig. 98) at which the tool is ground is as follows:

$$\tan \beta = \tan \phi \csc \omega. \qquad \sin \alpha = \cos \phi \cos \omega.$$

$$\sin \psi = \tan \phi \tan \alpha. \qquad d = 2H \sin \psi \tan \alpha.$$

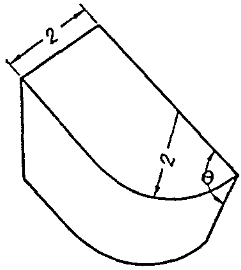
$$T = H \sin \psi \sin \alpha. \qquad W = \frac{(c + d)(c - d) \cos^2 \alpha}{4H \cos \psi}.$$

$$S = \frac{W + H \cos \psi}{2}. \qquad \tan \Sigma = \frac{T}{S}. \qquad r = S \sec \Sigma.$$

The radial tool must be placed on the adjustable angle plate so that the angle of rotation is $90° - \omega$ and the adjustable angle plate must be tilted at the angle β in order that the path of the grinding wheel be parallel to the axis MN. Figure

99 is given to assist the student in verifying the foregoing solution.

PROBLEMS



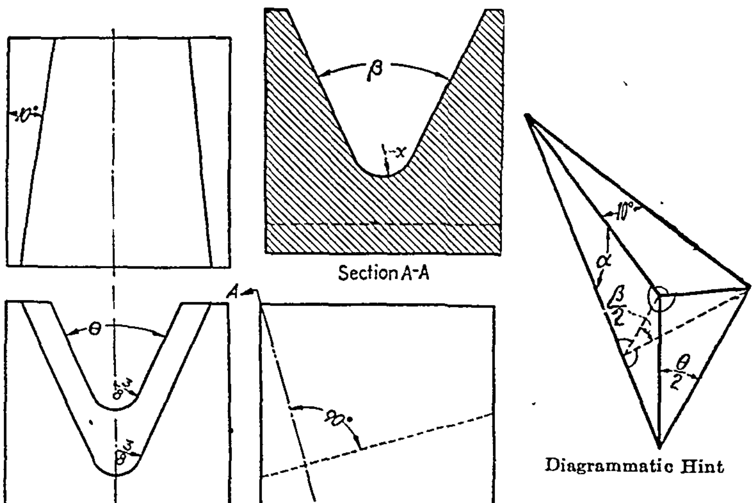
VARIABLE		
No.	Sym.	Value
1	θ	85°
2	θ	83°
3	θ	81°
4	θ	79°
5	θ	77°
6	θ	75°

$\theta = 73°$

Ans. $\begin{cases} \beta = 46° \, 16' \, 50'' \\ r = 1.9577 \end{cases}$

20. Determine the tilting angle β.

21. Determine the radius r at which the tool is to be ground.



$\theta = 36°$

Ans. $\begin{cases} \beta = 40° \, 34' \, 40'' \\ x = .39312 \end{cases}$

VARIABLE

1. $\theta = 48°$ **2.** $\theta = 46°$ **3.** $\theta = 44°$
4. $\theta = 42°$ **5.** $\theta = 40°$ **6.** $\theta = 38°$

22. Determine the angle β. **23.** Determine the radius x.

FLAT FORM TOOLS

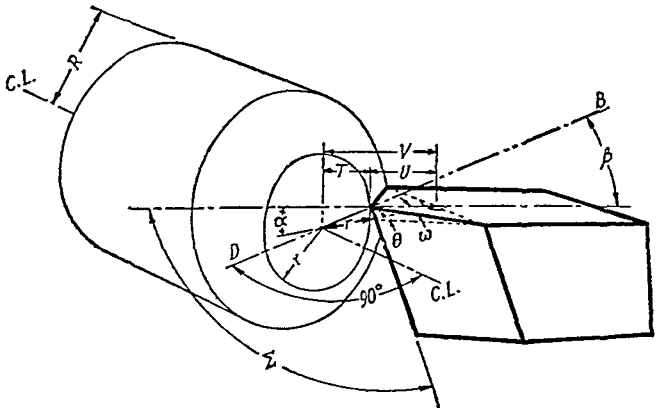


FIG. 100.

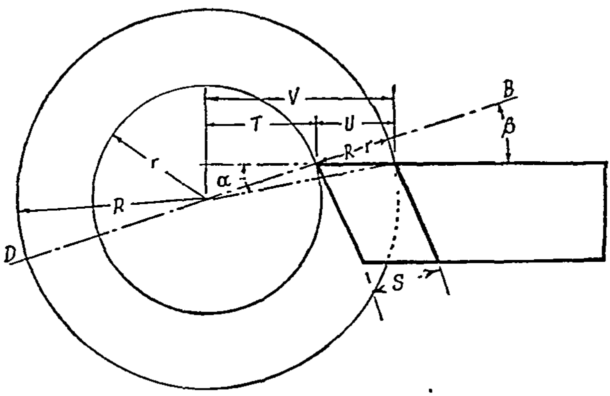


FIG. 101.

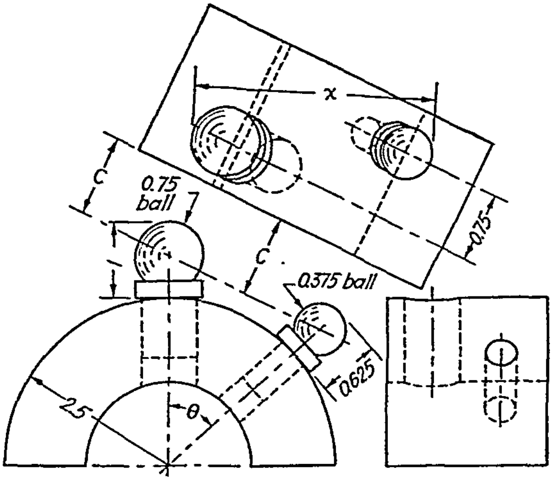
The path of the tool is along the line *BD*.

$$\beta = \text{top rake angle.}$$
$$\Sigma - 90^\circ = \text{clearance angle.}$$

From Figs. 100 and 101, verify the following formulas.

$$\sin \alpha = \frac{r \sin \beta}{R}, \qquad V = R \cos \alpha,$$

(Discussion continued on next page.)

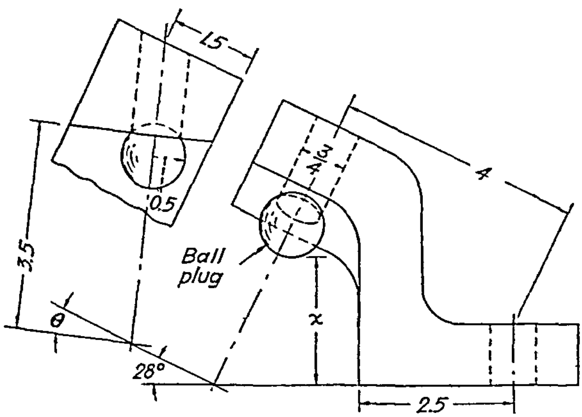


$\theta = 68°$
Ans. $x = 3.7658$

VARIABLE

1. $\theta = 62°$	**2.** $\theta = 63°$	**3.** $\theta = 64°$
4. $\theta = 65°$	**5.** $\theta = 66°$	**6.** $\theta = 67°$

26. Determine the distance x.



$\theta = 23°$
Ans. $x = 2.0758$

VARIABLE

1. $\theta = 17°$	**2.** $\theta = 18°$	**3.** $\theta = 19°$
4. $\theta = 20°$	**5.** $\theta = 21°$	**6.** $\theta = 22°$

27. Determine the distance x.

Also in a vertical plane passing through the point C:

$$E'D' = E'F' \sin 50^\circ. \qquad D'F' = E'F' \cos 50^\circ.$$
$$E'B' = .75 \cos 50^\circ. \qquad B'L = .75 \sin 50^\circ.$$
$$D'B' = E'B' - E'D'$$
$$F'K = 1.5 + EF - .4 - B'L - D'F'$$

The projection of the horizontal line $D'B'$ into the horizontal plane through F is equal to DB.

$$\therefore D'B' = DB.$$

In the horizontal plane through F:

$$CB = .8. \qquad CD = .8 - DB.$$
$$\angle JCD = 180^\circ - (90^\circ - 54^\circ) = 144^\circ.$$

In $\triangle JCD$ solve for the distance JD.

Consider JD in the horizontal plane passing through F.

DF' is in both the vertical plane through C and the vertical auxiliary plane AA. See diagram below.

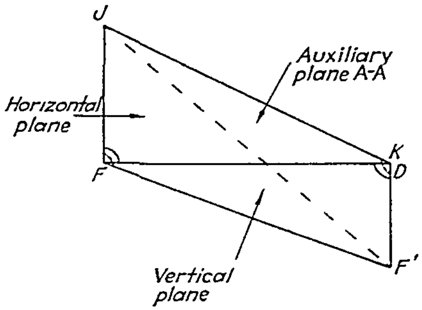
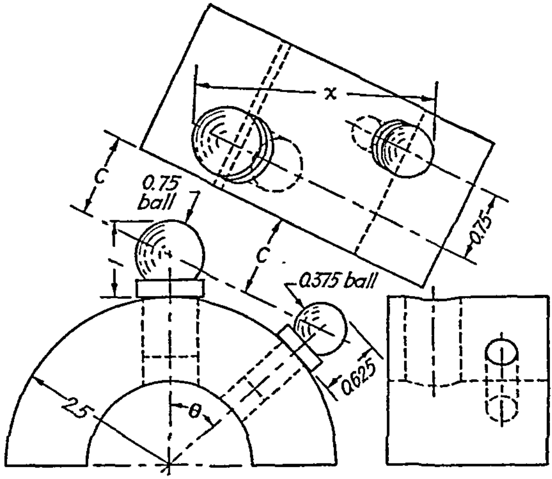


FIG. 104.

In $\triangle JDF'$, which lies in the vertical auxiliary plane AA, solve for the distance JF'.

Finally $X = JF' + .6 + .6$

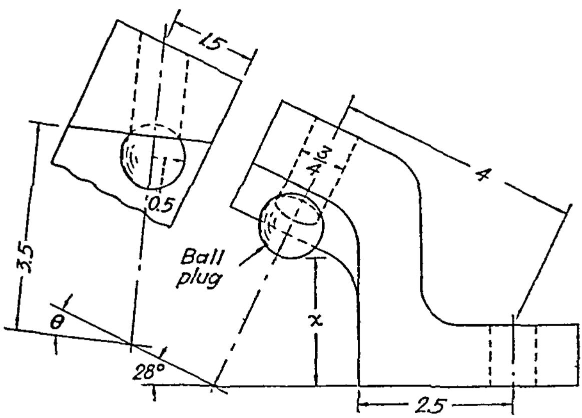


$\theta = 68°$
Ans. $x = 3.7658$

VARIABLE

1. $\theta = 62°$	**2.** $\theta = 63°$	**3.** $\theta = 64°$
4. $\theta = 65°$	**5.** $\theta = 66°$	**6.** $\theta = 67°$

26. Determine the distance x.

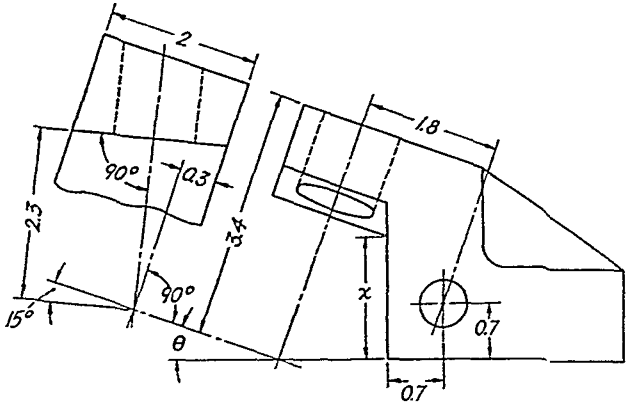


$\theta = 23°$
Ans. $x = 2.0758$

VARIABLE

1. $\theta = 17°$	**2.** $\theta = 18°$	**3.** $\theta = 19°$
4. $\theta = 20°$	**5.** $\theta = 21°$	**6.** $\theta = 22°$

27. Determine the distance x.

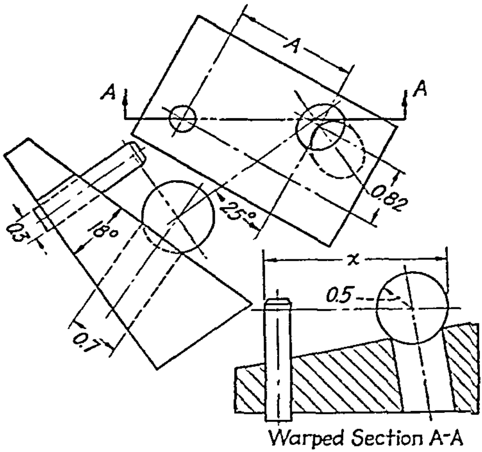


$\theta = 22°$
Ans. $x = 1.4608$

VARIABLE

1. $\theta = 16°$ **2.** $\theta = 17°$ **3.** $\theta = 18°$
4. $\theta = 19°$ **5.** $\theta = 20°$ **6.** $\theta = 21°$

28. *Determine the distance x.*



$A = 2.1$
Ans. $x = 2.8295$

VARIABLE

1. $A = 1.5$ **2.** $A = 1.6$ **3.** $A = 1.7$
4. $A = 1.8$ **5.** $A = 1.9$ **6.** $A = 2.0$

29. Determine the distance x.

CHAPTER IV

SCREW THREADS

Definitions

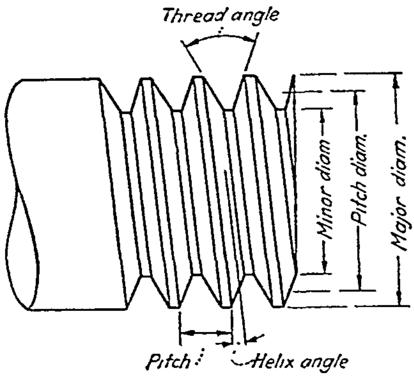


FIG. 105.

A screw **thread** is the projecting helical rib of a screw wound around a cylinder a definite number of times per inch. Sometimes, thread screws are made with two or more such helical ribs which are equally spaced, and are called double threaded, triple threaded, etc.

The **pitch** of a thread screw is the distance between two successive corresponding points (*e.g.*, the distance between two successive peaks).

The **number of threads per inch** is the number of peaks per inch. Hence the pitch is the reciprocal of the number of threads per inch.

The **lead** is the distance that a thread advances along a line parallel to the axis in one complete revolution. For a single-threaded screw, the lead is equal to the pitch; for a double-threaded screw, the lead is equal to twice the pitch; etc.

The **pitch diameter** is the diameter of a cylinder around

which the helical screw thread is wound, the width of the thread cut on the surface of this cylinder being equal to the width of the space.

The **helix angle** is the angle made by the tangent to the helical thread at the extremity of a pitch diameter and the plane through that point perpendicular to the axis. If a screw thread is rolled along a plane, as in Figs. 106 and 107, the angle between the straight parallel lines thus formed by the impression marks of the thread and a line perpendicular to the axis is also the helix angle.

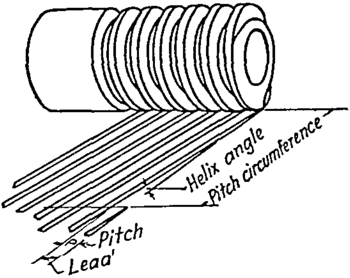


FIG. 106.

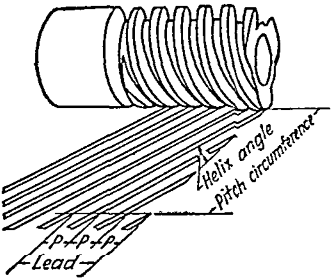


FIG. 107.

From the foregoing figures the tangent of the helix angle is:

$$\tan \beta = \frac{\text{lead}}{\pi \times \text{pitch diam.}} \text{ or } \frac{.3183 \times \text{lead}}{\text{pitch diam.}}.$$

The **major diameter** of a thread screw is the outside diameter of the thread screw.

The **minor diameter** of a thread screw is the root diameter or minimum diameter of the thread screw.

The **depth of thread** is one-half the difference between the major and minor diameters, *i.e.*, the distance between a crest and root measured perpendicular to the axis.

The **angle of thread** is the angle between the sides of the thread measured in a plane passing through the axis.

TYPES OF SCREW THREADS

The six commonest types of screw threads are: American ational, United States V, Square, Acme, Worm, and Buttress.

The principal characteristics of each of these threads will now be described and diagrams and formulas given for each.

NOTATION

N = number of threads per inch.
L = lead of thread.
d = depth of thread.
F = width at bottom of thread.
D = outside diameter of screw.
β = helix angle of thread.
n = number of threads wound around a screw.
P = distance between thread centers (called pitch).
W = diameter of wire used to check thread.
M = expansion of micrometer over three plugs.
S = pitch diameter of screw.

Formulas for Screw Threads

No.	To find	Formula
1	Number of threads per inch	$N = \frac{1}{P}$
2	Number of threads per inch	$N = \frac{n}{L}$
3	Lead of thread	$L = nP$
4	Lead of thread	$L = \frac{n}{N}$
5	Distance between thread centers	$P = \frac{L}{n}$
6	Distance between thread centers	$P = \frac{1}{N}$
7	Tangent of helix angle β	$\tan \beta = \frac{.3183L}{S}$

American National Thread

The screw thread most often used is the American National thread. This thread is divided into two classes called the coarse and the fine.

The American National coarse-thread series is recommended for general use in engineering work, in machine construction

where conditions are favorable to the use of bolts, screws, and other components where quick and easy assembly of the parts is desired, and for all work where conditions do not require the use of fine pitch threads.

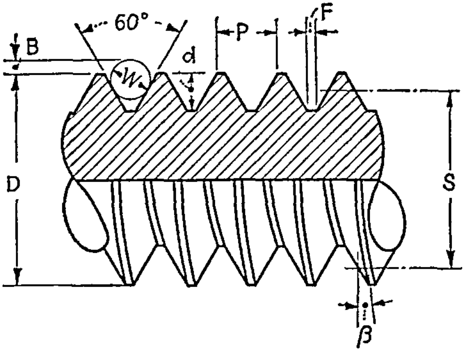


Fig. 108.
$B = 1.5W - 0.7578P$, $F = 0.125P$
$d = 0.6495P$, $S = D - d$.

The American National fine-thread series is recommended for general use in automotive and aircraft work, for use where the design requires both strength and the reduction in weight, and where special conditions require a fine thread.

Each classification of the American National thread embodies four different fits, *viz.*, loose fit, free fit, medium fit, and close fit. The loose fit (Class 1) includes screw-thread work of rough commercial quality, when the threads must assemble readily, and a certain amount of shake or play is not objectionable. Free fit (Class 2) includes the great bulk of screw-thread work of ordinary quality of finished and semi-finished bolts and nuts, machine screws, etc. Medium fit (Class 3) includes the better grades of interchangeable screw-thread work. Close fit (Class 4) includes screw-thread work requiring a fine snug fit, somewhat closer than the medium fit. In this class of fit, selective assembly of parts may be necessary. The quality of fit depends upon the relative size and the quality of finish of the mating parts. The notation used to indicate a threaded part 1 in. in diameter, 8 threads per

inch, and a Class 1 fit of the American National coarse threads is as follows: 1″ – 8 – NC – 1. The notation used to indicate a threaded part 1 in. diameter, 14 threads per inch, left-hand thread, and a Class 4 fit of American National fine thread is as follows: 1″ – 14LH – NF – 4. For screw threads of the American National form but of special pitches and special diameters, the symbol *NS* is used. The notation used to indicate threads of the American National form (pitch not standard) 1 in. diameter, 12 threads per inch, left-hand thread with a Class 4 fit is as follows: 1″ – 12LH – NS – 4. It must be remembered that the number of threads per inch is always stated regardless of whether or not it is the standard number of threads for that particular size. No symbol is used to distinguish right-hand threads.

The United States V Thread

The United States V thread is frequently used on coarse pitches. It has the same angle of thread as the American National but has no flat at the top and bottom of the thread.

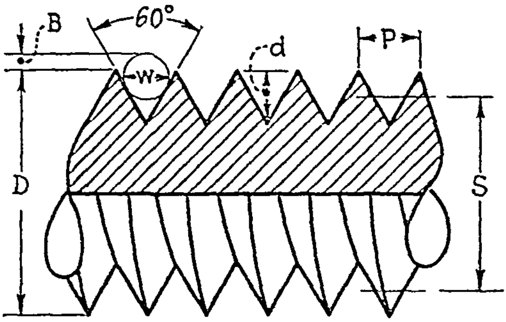


FIG. 109.

$B = 1.5W - 0.866P,$
$d = 0.866P, \qquad S = D - 0.866P.$

Square Thread

The Square thread is used on lead screws and is exactly what the name typifies, a cross-sectional view showing the helical rib to be a perfect square. The sides of the thread are parallel, and the depth is equal to the space between the threads and therefore equal to one-half the pitch. The space between the

threads in the nut is usually made a little oversize to allow screw to slide freely.

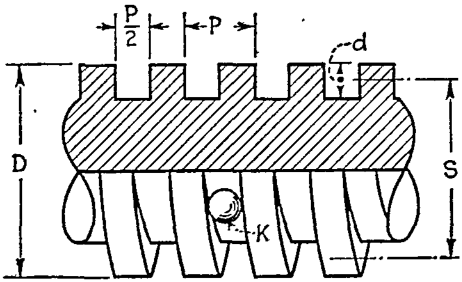


FIG. 110.

$$\text{Diameter of ball } K = \frac{P \cos \beta}{2}.$$

$$S = D - \frac{P}{2}.$$

Acme Thread

The Acme thread is also used on lead screws; in fact it is to be preferred to the square thread, because it acts more smoothly and does not have the backlash so often found on square threads. The angle of an Acme thread is 29°.

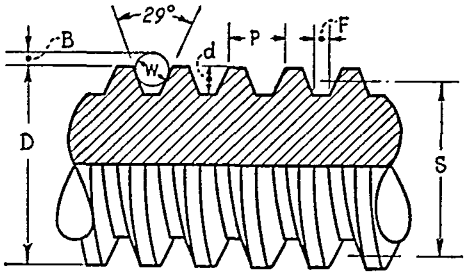


FIG. 111.

$$B = 2.5\ (W - 0.4873\ P \cos \beta), \qquad d = \frac{P}{2} + 0.01,$$

$$F = 0.3705\ P - 0.005, \qquad S = D - \frac{P}{2}.$$

A suggestion for obtaining the foregoing equation for B as follows:

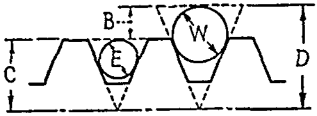


FIG. 112.

$$\frac{D}{C} = \frac{W}{E}.$$

Proportion by division,

$$\frac{D - C}{C} = \frac{W - E}{E}$$

or

$$D - C = \frac{C}{E}(W - E).$$

where $D - C = B$ and $\frac{C}{E}$ may be shown to be 2.5 approximately, where E is the diameter of the ball perpendicular to the thread and flush with outside diameter of thread.

Worm Thread

The Worm thread is used only in worm gearing. The angle of thread is 29°.

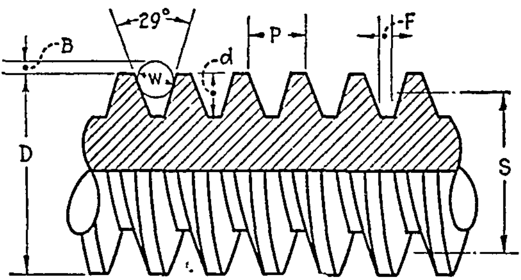


FIG. 113.

$B = 2.5\ (W - 0.5149\ P \cos\beta)$, $F = 0.31P$,
$d = 0.6866P$, $S = D - 0.6366P$.

Buttress Thread

The Buttress thread is used chiefly for heavy duty such as in a powerful jack screw, etc. The angle of thread is 45° where one side of the thread is at right angles to the axis of the thread.

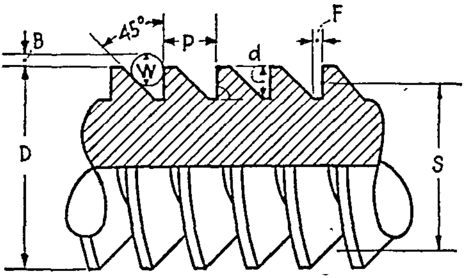
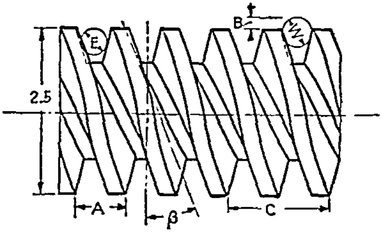


FIG. 114.

$F = 0.125P - 0.010,\ d = 0.75P + 0.010$
$S = D - 0.75P$

PROBLEMS

Double Thread Worm

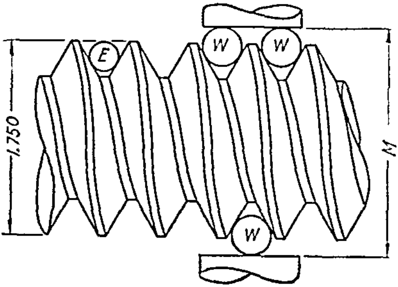


DOUBLE VARIABLE

No.	Sym.	Value	Sym.	Value
1	A	.250	W	.152
2	A	.3125	W	.199
3	A	.375	W	.238
4	A	.4375	W	.282
5	A	.5	W	.325
6	A	.5625	W	.368

1. Is C equal to the pitch, or the lead?
2. Determine the value of C.
3. Determine the value of E.
4. Determine the value of B.
5. Determine the helix angle β.

American National Thread



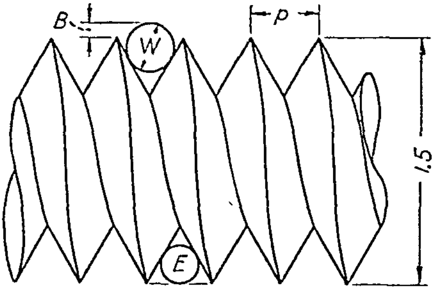
DOUBLE VARIABLE

No.	Sym.	Value	Sym.	Value
1	F	4	W	.198
2	F	5	W	.155
3	F	6	W	.125
4	F	7	W	.104
5	F	8	W	.089
6	F	9	W	.077

F threads per inch. Single thread.
W = diameter of wire.

6. Determine the diameter of E.
7. Determine the distance M.

United States Sharp V Thread

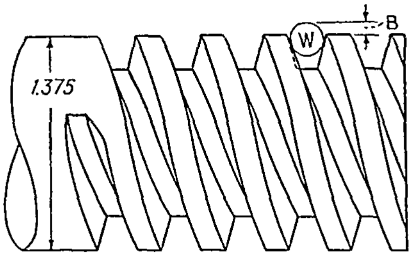


		DOUBLE	VARIABLE		
No.	Sym.	Value	Sym.	Value	
1	P	.5625	W	.532	
2	P	.5000	W	.470	
3	P	.4375	W	.407	
4	P	.3750	W	.345	
5	P	.3125	W	.282	
6	P	.2500	W	.220	

W = diameter of wire

8. Determine the number of threads per in.
9. Determine the diameter of E.
10. Determine the distance B.

Acme Thread

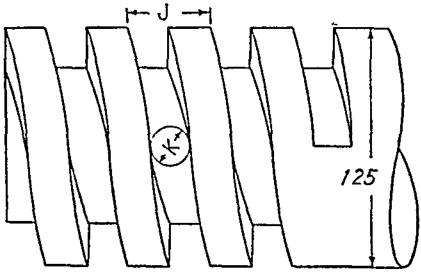


		DOUBLE	VARIABLE	
No.	Sym.	Value	Sym.	Value
1	H	12	W	.048
2	H	14	W	.039
3	H	16	W	.034
4	H	18	W	.027
5	H	8	W	.073
6	H	10	W	.057

H threads per inch. Double thread.
W = diameter of wire.

11. Determine the lead of thread.
12. Determine the helix angle β.
13. Determine the distance B.

Square Thread (Single)



	VARIABLE	
No.	Sym.	Value
1	J	.731
2	J	.762
3	J	.793
4	J	.824
5	J	.855
6	J	.886

14. Determine the diameter of a ball that will just fit in groove of thread.
15. Determine the helix angle β.

CHAPTER V

GEARS

SPUR GEARS

Gears are used to transmit power from one rotating shaft to another by means of intermeshing teeth which prevent loss of power due to slippage which would occur if power were transmitted by two disks in contact. The teeth are constructed on the circumferences of disks by alternately adding lugs and cutting recesses as indicated in Fig. 115.

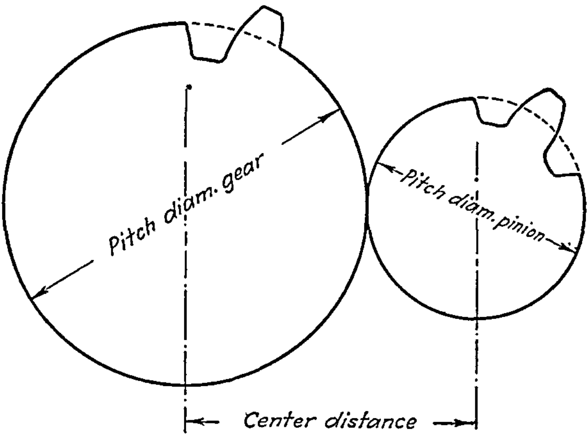


Fig. 115.

The circumferences of these disks are called the **pitch circles** of the gears, and the diameters are called the **pitch diameters.** The distance between the centers of the two pitch circles is called the **center distance.** Great care should be taken in calculating the pitch diameter of a gear since the teeth are constructed on the pitch circumference. In fact,

everything concerning the tooth is based upon the pitch circumference.

The **addendum** is the distance that the tooth extends above the pitch circumference (see Fig. 118).

The **dedendum** is the distance that the tooth falls below the pitch circumference.

The **working depth** is equal to two addendums, one of which is considered above and the other below the pitch circumference. In order to prevent the teeth from "bottoming," the dedendum is made greater than the addendum, the excess of the dedendum over the addendum being called the **clearance.**

The **diametral pitch** of a gear is equal to the number of teeth within π in. of the pitch circumference which is the same as the number of teeth within π in. along a straight line. The diametral pitch states the size of the tooth, in the same manner that the number of threads per inch of a screw states the size of the thread. A 5-diametral pitch gear is one that has five teeth within π in. of the pitch circumference, a 12-diametral pitch gear is one that has 12 teeth within π in. of the pitch circumference, etc. It is thus evident that, as the diametral pitch increases, the size of the tooth decreases accordingly. In order that two gears will mesh, they must have the same diametral pitch. The diametral pitches start with one-half and vary by quarters up to 3, then $3\frac{1}{2}$, and from 4 to 20 varying by 1, and from 20 up varying by 2.

The **circular pitch** is the distance from the center of one tooth to the center of the next, measured along the pitch circumference. It follows that the circular pitch is equal to π divided by the diametral pitch. Circular pitch is nearly always given in multiples of sixteenths of an inch expressed decimally. Thus: .1875, .250, etc.

In a pair of gears, the one having the greater number of teeth is called the **gear**, and the one having the lesser number is called the **pinion.**

The terms defined above are very essential to the solution of any spur-gear problem and therefore must be thoroughly learned. In particular, the student is warned not to confuse the meaning of diametral pitch with pitch diameter.

In order that the gears rotate uniformly, the profile of the tooth must have some definite curvature. The most practical curvature and the one most commonly used is the involute curve. The involute is a curve traced by a point on an inextensible piece of string which is unwound from a circular disk, the circumference of which is called the base circle of the gear.

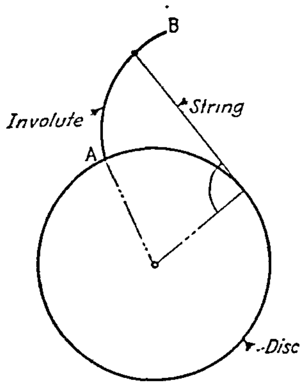


FIG. 116.

In Fig. 116, the point A which is the starting point of a piece of string being unwound from the circular disk moves along the curve AB which is called the involute.

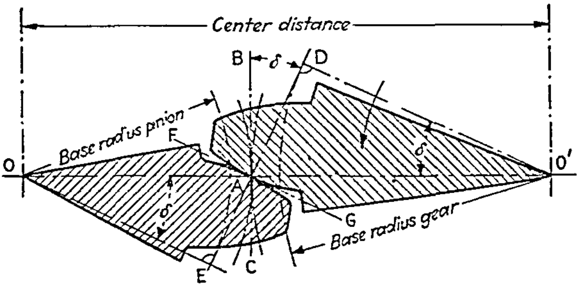


FIG. 117.

In Fig. 117, the right-hand tooth of a gear is caused to rotate in the counterclockwise direction as indicated. The force at point A caused by this rotating tooth is in the direction AC. However, the force exerted by this tooth, causing the

left-hand tooth to rotate, is in the direction AE which is at right angles to the line of contact FG of the teeth. The angle δ between the two lines of forces BC and DE is called the pressure angle. Gears were found by experiment to be most efficient when the pressure angle is $14\frac{1}{2}°$. For this pressure angle the outward force tending to separate the axes is nearly neutralized by the friction between the teeth, and the thickness of the tooth at the base is great enough to prevent the teeth from breaking. In order to strengthen the teeth without using larger teeth, a special pressure angle of 20° and shorter teeth called stub teeth are sometimes used. The greater force tending to separate the axes caused by this larger pressure angle is taken care of by the use of better bearings.

Formulas for Spur Gears

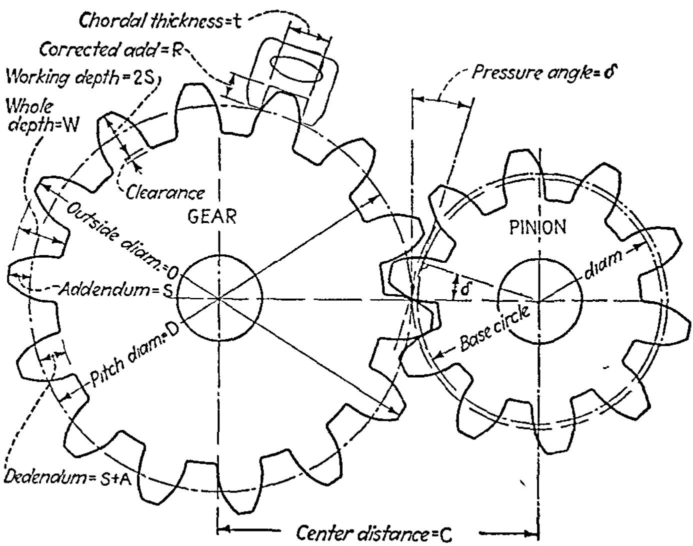


FIG. 118.

NOTATION

N = number of teeth in gear
P = diametral pitch
D = pitch diameter
A = clearance
O = outside diameter
S = addendum

n = number of teeth in pinion
P' = circular pitch
$S + A$ = dedendum
W = whole depth of tooth
C = center distance
δ = pressure angle

DIAMETRAL PITCH

No.	To find	Formula
1	Diametral pitch	$P = \frac{N + n}{2C}$
2	Diametral pitch	$P = \frac{N + 2}{O}$
3	Pitch diameter	$D = \frac{N}{P}$
4	Outside diameter	$O = \frac{N + 2}{P}$
5	Center distance	$C = \frac{N + n}{2P}$
6	Addendum	$S = \frac{1}{P}$
7	Dedendum	$S + A = \frac{1.157}{P}$
8	Clearance	$A = \frac{.157}{P}$
9	Whole depth of tooth	$W = \frac{2.157}{P}$
10	Circular pitch	$P' = \frac{3.1416}{P}$

CIRCULAR PITCH

No.	To find	Formula
11	Circular pitch	$P' = \frac{6.2832C}{N + n}$
12	Pitch diameter	$D = NP'(.3183)$
13	Outside diameter	$O = P'(N + 2).3183$
14	Center distance	$C = P'(N + n).1591$
15	Addendum	$S = P'(.3183)$
16	Dedendum	$S + A = P'(.3683)$
17	Clearance	$A = P'(.05)$
18	Whole depth of tooth	$W = P'(.6866)$
19	Diametral pitch	$P = \frac{3.1416}{P'}$

INTERNAL GEARS

No.	To find	Formulas
20	Center distance of internal gears	$C = \frac{N - n}{2P}$
21	Inside diameter of internal gears	$I = \frac{N - 2}{P}$
22	Diametral pitch of internal gears	$P = \frac{N - n}{2C}$

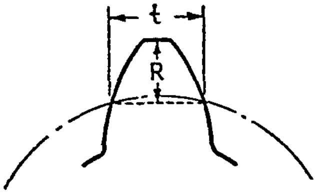


FIG. 119.

CHORDAL THICKNESS AND CORRECTED ADDENDUM CHART
(For diametral pitch = 1)

N	t	R	N	t	R
8	1.5607	1.0769	17	1.5686	1.0362
9	1.5628	1.0648	21	1.5694	1.0294
10	1.5643	1.0616	26	1.5698	1.0237
11	1.5654	1.0559	35	1.5702	1.0176
12	1.5663	1.0514	55	1.5706	1.0112
14	1.5675	1.0440	135	1.5708	1.0046

The **chordal thickness** is the thickness of a tooth at the pitch circle. Thus in Fig. 119, t is the chordal thickness.

The **corrected addendum** R is the perpendicular distance from the top of the tooth to the chord which represents the chordal thickness (Fig. 119).

All gear teeth must be checked for size by the measurement of the chordal thickness and the corrected addendum.

The chart accompanying Fig. 119 gives the chordal thickness and corrected addendum for a 1-diametral pitch and the

indicated number of teeth N. To find the chordal thickness and corrected addendum for gears of any other diametral pitch, divide the given values of t and R corresponding to the number of teeth in the chart nearest the number of teeth in the gear by the diametral pitch of the gear.

Example: Find the chordal thickness and corrected addendum for a 7-diametral pitch gear having 47 teeth.

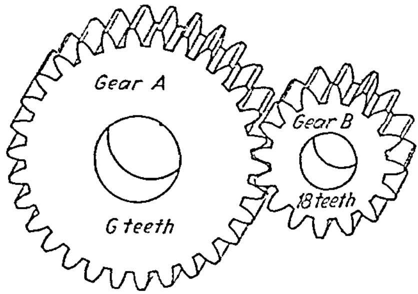
Solution: The nearest number of teeth in the chart to 47 is 55. Hence, the chordal thickness is $1.5706 \div 7 = .224$, and the corrected addendum is $1.0112 \div 7 = .144$.

CUTTER CHART

No. 1	for	135 teeth to rack	
No. $1\frac{1}{2}$	for	80 teeth to	134 teeth
No. 2	for	55 teeth to	79 teeth
No. $2\frac{1}{2}$	for	42 teeth to	54 teeth
No. 3	for	35 teeth to	41 teeth
No. $3\frac{1}{2}$	for	30 teeth to	34 teeth
No. 4	for	26 teeth to	29 teeth
No. $4\frac{1}{2}$	for	23 teeth to	25 teeth
No. 5	for	21 teeth to	22 teeth
No. $5\frac{1}{2}$	for	19 teeth to	20 teeth
No. 6	for	17 teeth to	18 teeth
No. $6\frac{1}{2}$	for	15 teeth to	16 teeth
No. 7	for	14 teeth	
No. $7\frac{1}{2}$	for	13 teeth	
No. 8	for	12 teeth	

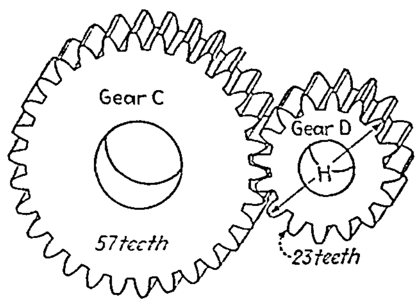
The chart above shows the number of cutter to be used for the different numbers of teeth in the gear. The numbers of the cutter are arbitrary and merely represent the curvatures on the sides of the teeth. When a spur gear is cut with a rotary cutter, it is necessary to know both the diametral pitch and the number of cutter to be used.

PROBLEMS



Diametral pitch = 6

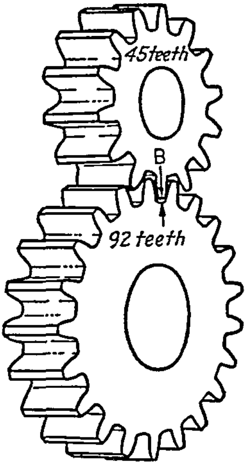
1. Determine the addendum.
2. Determine the outside diameter of gear *A*.



3. Determine the chordal thickness of gear *D*.
4. Determine the corrected addendum of gear *D*.
5. Determine the diametral pitch.

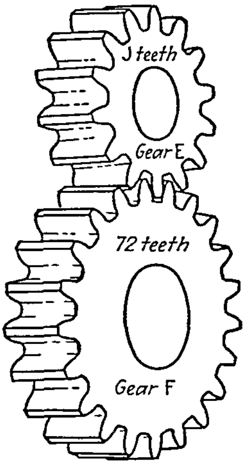
VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	*G*	22	24	26	28	30	32
2	*G*	22	24	26	28	30	32
3	*H*	5	6.25	3.125	2.5	12.5	1.5625
4	*H*	5	6.25	3.125	2.5	12.5	1.5625
5	*H*	5	6.25	3.125	2.5	12.5	1.5625



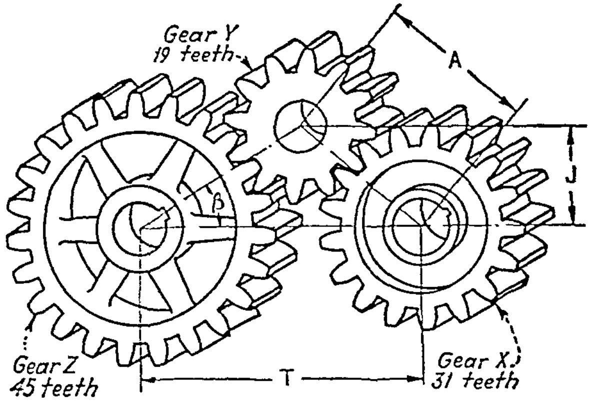
Diametral pitch = S = variable.

6. Determine the center distance.
7. Determine the clearance B.



Gears E and F are 7-diametral pitch.

8. Determine the whole depth of a tooth.
9. Determine the pitch diameter of gear E.
10. Determine the outside diameter of gear E.
11. Determine the pitch diameter of gear F.
12. Determine the number cutter for gear F.



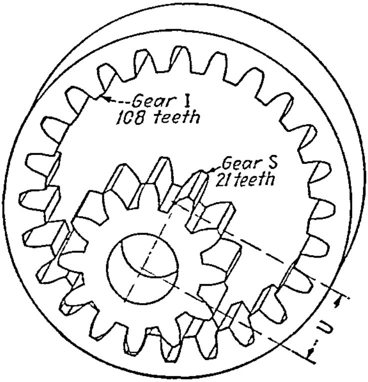
For problems 13 to 21, variable = A.
For problems 22 to 23, variables = A and T.

13. Determine the chordal thickness of tooth of gear Z.
14. Determine the chordal thickness of tooth of gear Y.
15. Determine the center distance of gears Y and Z
16. Determine the outside diameter of gear X.
17. Determine the pitch diameter of gear Y.
18. Determine the diametral pitch of gears X, Y, and Z.
19. Determine the corrected addendum of gear X.
20. Determine the outside diameter of gear Y.
21. Determine the outside diameter of gear Z.
22. Determine the angle β.
23. Determine the distance J.

VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
6	S	5	7	10	12	14	16
7	S	5	7	10	12	14	16
8	J	24	27	31	36	38	43
9	J	24	27	31	36	38	43
10	J	24	27	31	36	38	43
11	J	24	27	31	36	38	43
12	J	24	27	31	36	38	43
13–23	A	4.1666	2.7778	3.1250	2.0833	3.5714	5.0000
22–23	T	7.25	5.00	5.50	4.25	6.25	8.50

An internal gear is one having its teeth extending toward the center of the pitch circle as in the following figure:



Variable		
No.	Sym.	Value
1	U	2.5588
2	U	2.7187
3	U	3.6250
4	U	2.4166
5	U	4.3500
6	U	3.1071

Center distance $= U =$ variable.

See page 156 for formulas on internal gears.

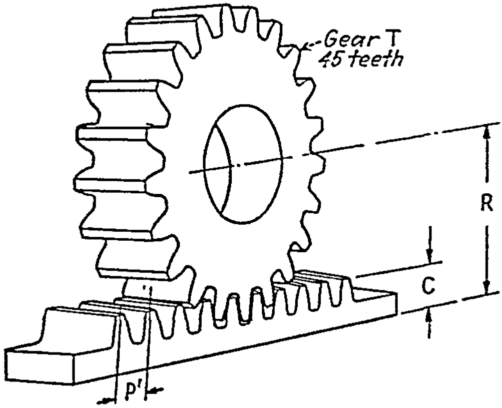
24. Determine the diametral pitch of the internal gear *I*.

25. Determine the inside diameter of gear *I*.

26. Determine the pitch diameter of gear *S*.

A rack consists of teeth on the surface of a rectangular piece along which a gear may roll. The center distance between the teeth of a rack is equal to the circular pitch of the mating gear, and the total depth of the teeth of the rack is equal to the total depth of the teeth of the gear.

The chordal thickness of the tooth of a rack is equal to one-half the circular thickness of the mating gear, and the "corrected addendum" of the tooth of a rack is equal to the addendum of the tooth of the mating gear.



Variable		
No.	Sym.	Value
1	R	3 301
2	R	3 312
3	R	3.263
4	R	3 234
5	R	3.225
6	R	3 296

Gear T and rack are 8-diametral pitch.

27. Determine the circular pitch P'.
28. Determine the outside diameter of gear T.
29. Determine the distance from the pitch line of rack to the base.
30. Determine the thickness C of the rack.

STUB-TOOTH GEARS

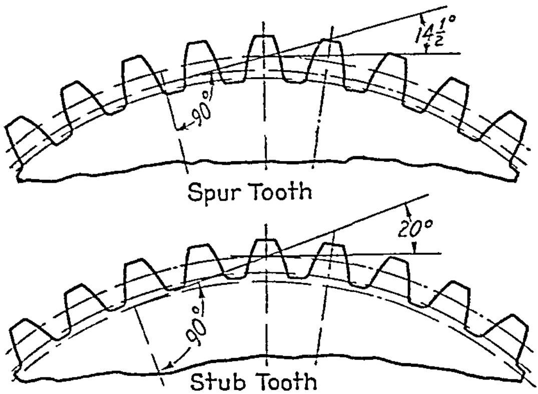


Fig. 120.

As previously mentioned the addendums and dedendums of a stub-tooth gear are shorter than those of the standard type, and the pressure angle is 20° instead of $14\frac{1}{2}°$ as shown in Fig. 120. The stub tooth is described by two diametral pitches given in the form of a fraction which is called the pitch of the stub tooth. The numerator of the fraction indicates the diametral pitch used in calculating the pitch diameter and

the thickness of the tooth. The denominator indicates the diametral pitch used in calculating the addendum, dedendum, and whole depth of the tooth. The numerator is always less than the denominator, which means that the diametral pitch governing the thickness of the tooth is less than that governing the addendum, dedendum, and whole depth of tooth, which in turn means that the outside diameter of the stub-tooth gear is less than that for the standard gear having the same pitch circle.

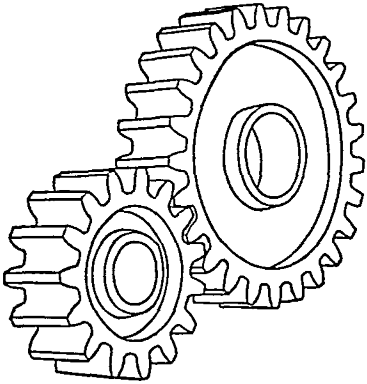
NOTATION

R_s = corrected addendum for stub tooth.
R = corrected addendum obtained by using the diametral pitch expressed in the numerator of the fraction.
P = diametral pitch expressed in the numerator of fraction.
P_s = diametral pitch expressed in denominator of fraction.
t_s = chordal thickness.

Formulas for Stub-tooth Gears

No.	To find	Formula
23	Diametral pitch	$P = \frac{N + n}{2C}$
24	Diametral pitch (for depth of tooth)	$P_s = \frac{2P}{O_s \times P - N}$
25	Pitch diameter	$D_s = \frac{N}{P}$
26	Outside diameter	$O_s = D_s + \frac{2}{P_s}$
27	Center distance	$C_s = \frac{N + n}{2P}$
28	Addendum	$S_s = \frac{1}{P_s}$
29	Dedendum	$(S + A)_s = \frac{1.157}{P_s}$
30	Whole depth of tooth	$W_s = \frac{2.157}{P_s}$
31	Corrected addendum	$R_s = R - \left(\frac{1}{P} - \frac{1}{P_s}\right)$
	The chordal thickness t_s of a stub tooth is the same as the chordal thickness for a standard tooth using the diametral pitch given in the numerator of the fraction.	

PROBLEMS



Variable		
No.	Sym.	Value
1	P	5/6
2	P	6/8
3	P	7/9
4	P	8/10
5	P	10/12
6	P	12/14

The gear has 25 teeth and the pinion 15 teeth and the pressure angle is 20°. The variable P is the pitch of the stub-tooth gears.

Determine:

1. The pitch diameter of gear.
2. The outside diameter of pinion
3. The center distance.
4. The whole depth of tooth.
5. The chordal thickness of gear tooth.
6. The corrected addendum of pinion tooth.

DUPLICATION OF SPUR GEARS

When gears wear out or a tooth breaks, it becomes necessary to duplicate the pair of gears. The two factors which determine the size of the gear and tooth are the number of teeth and the pitch (either diametral or circular). The number of teeth can, of course, be counted, which leaves only the pitch to be determined. The simplest method of determining the diametral pitch is to put bluing on the tops of the teeth and roll the gear along a straight line on a piece of paper. Count the number of spaces between impressions in π in. This will always be a whole number and a fraction. The diametral pitch will be the next possible value ($\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{2}$, 4, 5, etc., to 20, 22, 24, etc.) above this counted value. Thus 2.05 spaces means a diametral

pitch of $2\frac{1}{4}$; 5.9 spaces means a diametral pitch of 6. The number of spaces is always less than the diametral pitch because the diametral pitch should be measured on the pitch circle instead of along the outside circumference. Verify the diametral pitch thus obtained by computing the outside diameter by formula and checking with the measured outside diameter of gear. If these two values agree (within five thousandths), the diametral pitch thus obtained is correct.

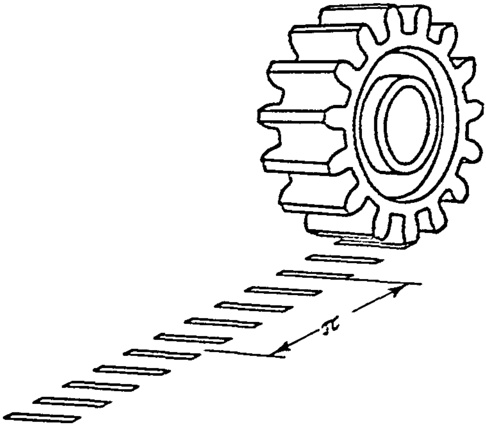


FIG. 121.

Example a: In the foregoing figure the measured outside diameter is 3.395 and the number of spaces in π in. is about 4.7, which means that the diametral pitch is probably 5. To verify, compute the outside diameter by formula 4. Thus: $O = \frac{15 + 2}{5} = 3.4$ which checks with the measured outside diameter of 3.395.

If the computed outside diameter does not check with the measured value, the gear is either a stub-tooth gear or it has a circular pitch instead of a diametral pitch. In this case, measure the center distance in the machine occupied by these gears. Then compute the diametral pitch by formula 1. If one of the possible values of the diametral pitch is thus obtained

(within .005), the gear is a stub-tooth gear, and this value is the numerator of the pitch of the stub-tooth gear. Some of the common values for the pitch of stub-tooth gears are $\frac{4}{5}$, $\frac{5}{7}$, $\frac{6}{8}$, $\frac{7}{9}$, $\frac{8}{10}$, $\frac{9}{11}$, $\frac{10}{12}$, $\frac{12}{14}$, $\frac{14}{16}$, etc., to $\frac{20}{22}$.

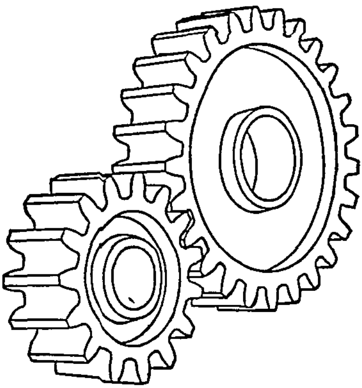


FIG. 122.

Example b: In the gears of Fig. 122, consider that the method of rolling leads to a computed value of the outside diameter which is considerably greater than the measured outside diameter. The measured center distance is 2.857, and the outside measured diameter of the pinion is 2.363. Using *formula 1,* $P = \frac{25 + 15}{2 \times 2.857} = 6.999$ *or a diametral pitch of 7.* This shows that the gears have stub teeth, and that 7 is the numerator of the pitch. From the possible values of stub pitches, the denominator must be 9. This will now be checked by computing the outside diameter of the pinion by the formula $O = \frac{N}{P} + \frac{2}{P_s} = \frac{15}{7} + \frac{2}{9} = 2.365$ as compared with measured value of 2.363.

If the rolling method does not lead to a satisfactory check, and if the foregoing method, using formula 1, gives a decimal value not close to one of the diametral pitches, the pitch of the gear must then be a circular pitch.

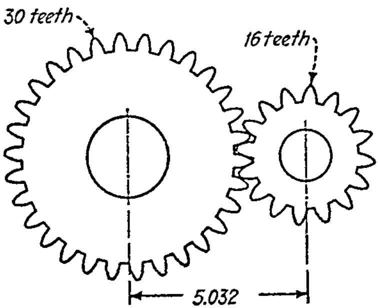


FIG. 123.

Circular pitch $= \frac{11}{16}$.

Example c: In the problem of Fig. 123, assume that the rolling method has been applied and fails to check.

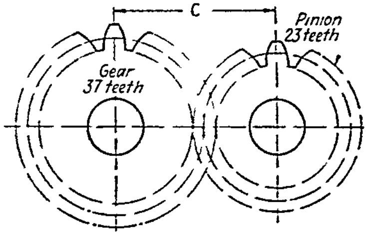
By applying formula 1, $P = \dfrac{N + n}{2C} = \dfrac{30 + 16}{2 \times 5.032} = \dfrac{46}{10.064}$ $= 4.5707$.

This is not a possible value of a diametral pitch so this is not a stub-tooth gear. Hence the pitch of this gear is undoubtedly expressed in circular pitch. Using formula 10, $P' = \dfrac{3.1416}{4.5707}$ $= .6873$ which is close to the possible circular pitch of .6875. Calculate the outside diameter of the gear by formula 13.

$$O = P'(N + 2).3183 = .6875 \times (30 + 2).3183 \doteq 7.003.$$

This value checks closely with the measured outside diameter.

PROBLEMS



VARIABLE		
No.	Sym.	Value
1	P'	.25
2	P'	3125
3	P'	.375
4	P'	.4375
5	P'	.5
6	P'	.5625

Circular pitch $= P' =$ variable.

1. Determine the pitch diameter of the gear.
2. Determine the outside diameter of the pinion.

3. Determine the chordal thickness of the gear.
4. Determine the corrected addendum of the pinion.

RELATION BETWEEN THE PRESSURE ANGLE AND THE INCLUDED ANGLE FOR THE INVOLUTE CURVE

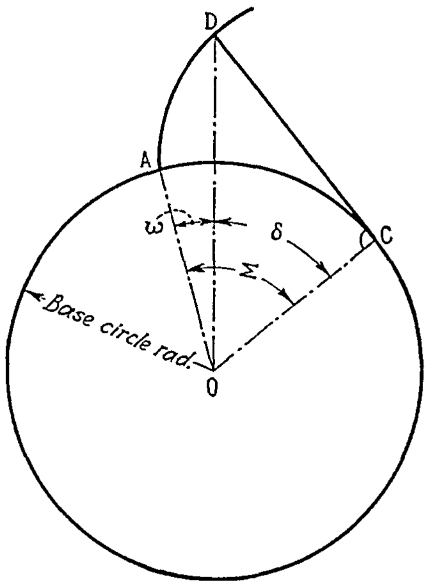


FIG. 124.

In checking gear teeth above and below the pitch circle and also in checking gear templets which are used for checking large gear teeth such as those used in a conveying system, two formulas are used which will now be derived. In the foregoing figure the angle Σ, which is the angle included by the radii OA and OC, will be called the included angle of the involute curve.

To prove:

$$\tan \delta = .017453\,\Sigma$$

and

$$\text{angle } \Sigma = 57.296 \tan \delta.$$

Proof: $\dfrac{\Sigma}{360°} = \dfrac{\widehat{AC}}{2\pi R}$ (Proposition 34, Chap. VI, Vol. I).

$$\widehat{AC} = \frac{2\pi R\Sigma}{360°} = \frac{\pi R\Sigma}{180°}.$$

$$\widehat{AC} = CD \text{ from definition of the involute.}$$

$$\tan\,\delta = \frac{CD}{OC} = \frac{\frac{\pi R\Sigma}{180°}}{R} = \frac{\pi R\Sigma}{180°} \times \frac{1}{R}$$

$$= \frac{\pi\Sigma}{180°} = .017453\,\Sigma.$$

olving for Σ,

$$\Sigma = \frac{\tan\,\delta}{.017453} = 57.296\,\tan\,\delta.$$

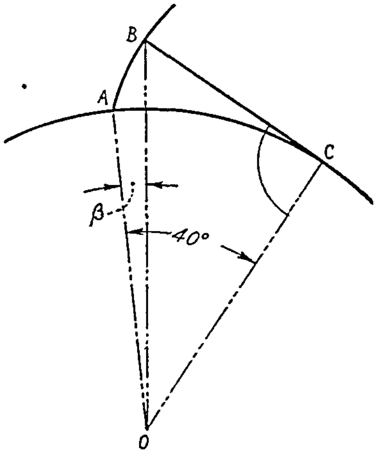


FIG. 125.

Example a: In Fig. 125, solve for β.
Solution: $\beta = 40° - \angle BOC$.
Using the formula,

$$\tan\,\delta = .017453\,\Sigma.$$
$$\tan\,\angle BOC = .017453 \times 40° = .69812.$$
$$\angle BOC = 34°\,55'\,10''.$$
$$\therefore \beta = 40° - 34°\,55'\,10'' = 5°\,4'\,50''.$$

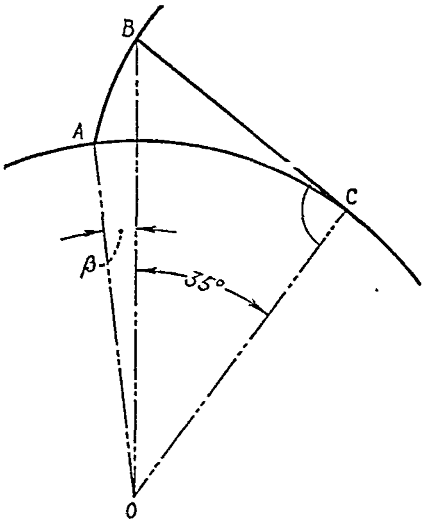


FIG. 126.

Example b: In Fig. 126, determine β.

Solution: $\beta = \angle AOC - 35°$

Using the formula, $\Sigma = 57.296 \tan \delta$.

$$\angle AOC = 57.296 \times .70021 = 40.1192°.$$

To change decimal degrees into minutes, multiply the decimal by 60. Multiply the remaining decimal by 60 to get the number of seconds.

Thus:

$$.1192° = .1192 \times 60' = 7'.152.$$
$$.152 \times 60 = 9''.$$

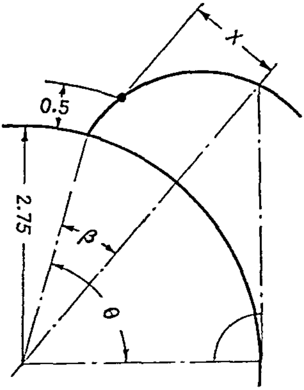
That is,

$$\angle AOC = 40.1192° = 40°\ 7'\ 9''.$$

Hence

$$\beta = 40°\ 7'\ 9'' - 35° = 5°\ 7'\ 9$$

PROBLEMS



No.	Sym.	Value
1	θ	55°
2	θ	57°
3	θ	59°
4	θ	61°
5	θ	63°
6	θ	65°

(Variable)

$\theta = 67°$

Ans. $\begin{cases} \beta = 17° \, 32' \, 43'' \\ x = .76765 \end{cases}$

1. Determine the angle β.
2. Determine the distance x.

CHECKING THICKNESS OF INVOLUTE TEETH ABOVE OR BELOW THE PITCH CIRCLE

In order to check the pressure angle at the pitch circle, the chordal thickness must be computed for given distances above or below the pitch circle. When checking below the pitch circle, the check can be made only as far as the base circle, because below that the curvature departs from the involute curve.

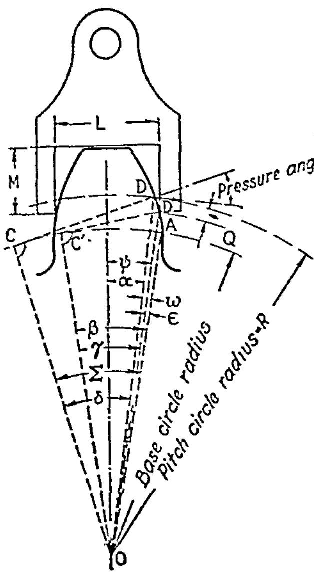


FIG. 127.

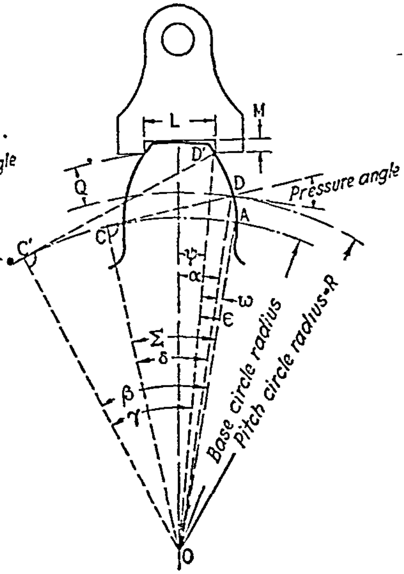


FIG. 128.

In Figs. 127 and 128, angle δ is equal to the pressure angle (sides respectively perpendicular) and angle Σ is the included angle of the involute curve; angle ω is the angle at the center of the base circle subtended by that portion of the involute curve from its origin to its intersection with the pitch circle.

Regardless of the number of teeth in the gear,

$$\begin{aligned}\omega &= \Sigma - \delta \\ &= 57.296 \tan \delta - \delta.\end{aligned}$$

For $\delta = 14\frac{1}{2}°$,

$$\begin{aligned}\omega &= 57.296 \times .25862 - 14.5° \\ &= 14.8179° - 14.5° = .3179° = 0° \, 19' \, 4''.\end{aligned}$$

Hence for a $14\frac{1}{2}°$ pressure angle, ω always is $0° \, 19' \, 4''$.

For $\delta = 20°$,

$$\begin{aligned}\omega &= 57.296 \times .36397 - 20° \\ &= 20.8540° - 20° = .8540° = 0° \, 51' \, 14''.\end{aligned}$$

$$x = \frac{N \sin \delta + (\pi S - .017453N\delta) \cos \delta}{P}$$

where

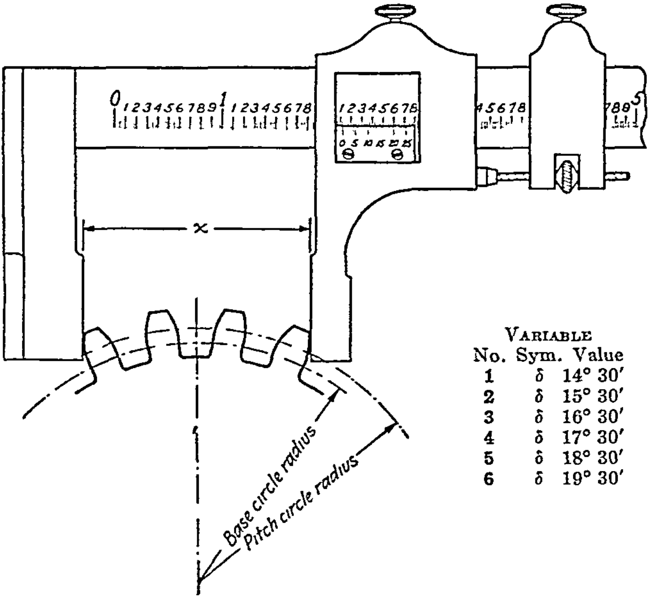
T = number of teeth within the Johansson blocks

and

$S = T - \frac{1}{2}.$

Note: The pressure angle δ within the parentheses must be expressed in decimal degrees.

PROBLEM



Variable		
No.	Sym.	Value
1	δ	14° 30′
2	δ	15° 30′
3	δ	16° 30′
4	δ	17° 30′
5	δ	18° 30′
6	δ	19° 30′

δ = 20° 30′
Ans. x = 3.5839

The above segment is of a spur gear having 30 teeth and a diametral pitch of 3.

1. Determine the distance x over four teeth.

CHECKING SPUR-GEAR TEETH BY MEANS OF PLUGS TANGENT AT THE PITCH CIRCUMFERENCE

If spur-gear teeth must be checked accurately, the best method is to use plugs. To get the best results, the plug must

be tangent to the teeth at the pitch circumference. The formulas for computing these distances are given below and should be verified by the student.

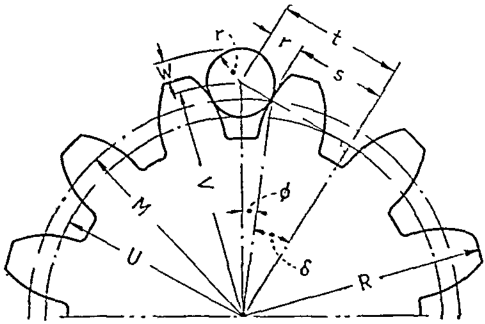


FIG. 130.

External Gears

$$U = M \cos \delta, \qquad \phi = \frac{90°}{N}, \qquad s = M \sin \delta$$
$$t = U \tan (\delta + \phi), \qquad V = U \sec (\delta + \phi)$$
$$r = t - s, \qquad W = V + r - R$$

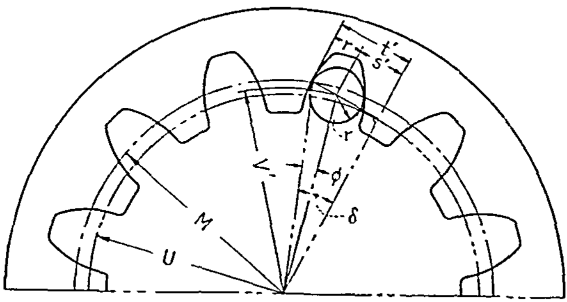


FIG. 131.

Internal Gears

$$U = M \cos \delta, \qquad \phi = \frac{90°}{N}$$
$$s' = U \tan (\delta - \phi), \qquad t' = M \sin \delta$$
$$V' = U \sec (\delta - \phi), \qquad r = t' - s'$$

CHECKING THE POSITION OF SPUR TEETH IN RELATION TO THE KEYWAY, OR CHECKING THE LOCATION OF THE TEETH IN TWO GEARS (FASTENED TOGETHER) WITH RESPECT TO ONE ANOTHER

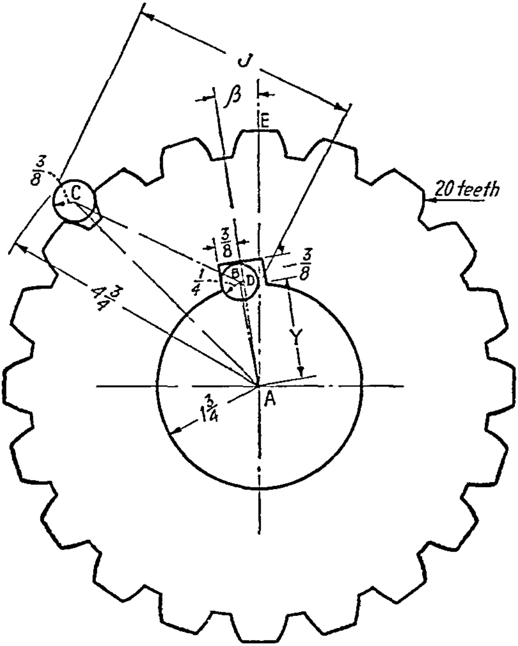


Fig. 132.
$J = 3.415$
Ans. $\beta = 17° \ 27' \ 15''$

In making spur-gear teeth, after the first cutting, the gear operator must check the location of the teeth with respect to the keyway. This is done by checking the distances CA, BA, and CB from which angle CAB may be computed by the cosine formula. Also angle BAD can be computed, and angle β is given on the blue print. The sum of these three angles must equal the known angle CAE. In case these angles do not check, the location of the teeth must be changed by rotating the cutter or the gear.

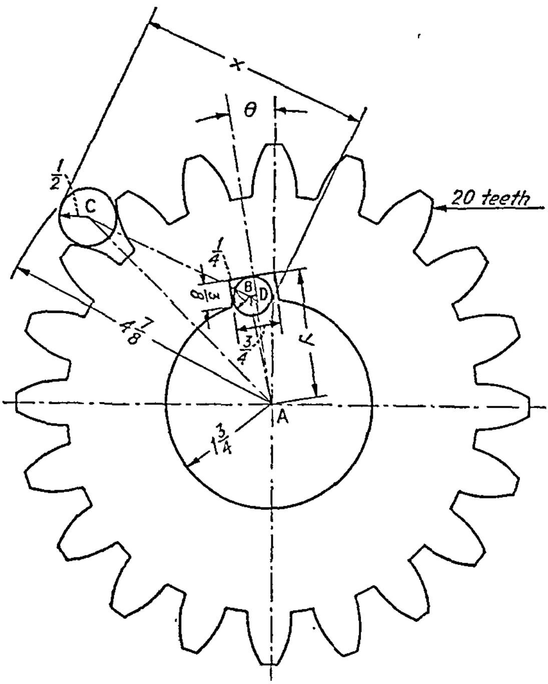
PROBLEMS

In Fig. 132, the variable is as follows:

VARIABLE

1. $J = 3.595$	2. $J = 3.565$	3. $J = 3.535$
4. $J = 3.505$	5. $J = 3.475$	6. $J = 3.445$

1. Determine the angle β.



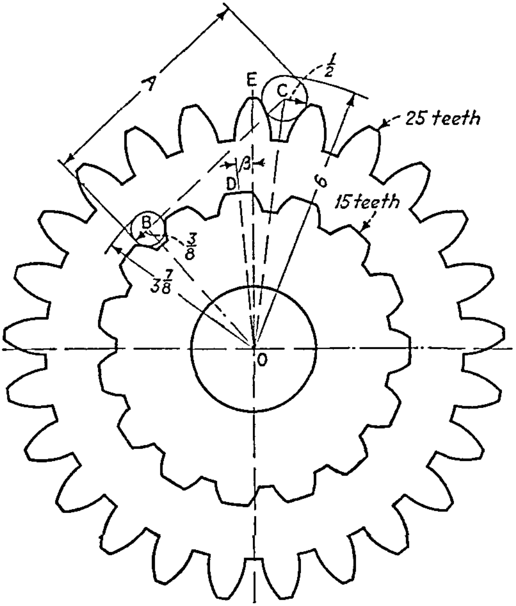
VARIABLE

No.	Sym.	Value
1	θ	2° 30′
2	θ	3° 30′
3	θ	4° 30′
4	θ	5° 30′
5	θ	6° 30′
6	θ	7° 30′

$\theta = 8° 30'$

Ans. $x = 3.7446$

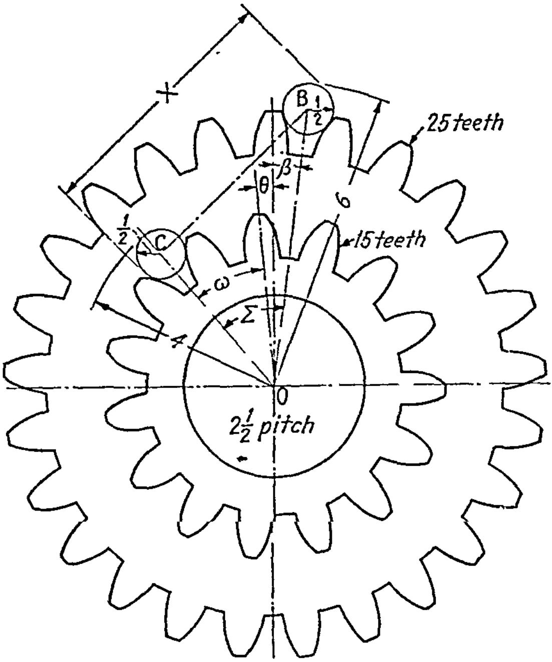
2. Determine the distance x.



$A = 5.042$
Ans. $\beta = 6° 2' 28''$

Variable		
No.	Sym.	Value
1	A	4.862
2	A	4.892
3	A	4.922
4	A	4.952
5	A	4.982
6	A	5.012

3. Determine the angle β.



Variable		
No.	Sym.	Value
1	θ	2° 15′
2	θ	2° 45′
3	θ	3° 15′
4	θ	3° 45′
5	θ	4° 15′
6	θ	4° 45′

$\theta = 5° 15'$
Ans. $x = 5.1187$

4. Determine the distance x.

BEVEL GEARS

Bevel gears are used to transmit power from one rotating shaft to another when the axes of the shafts intersect. The teeth are constructed on the lateral surfaces of two cones having their lateral surfaces in contact as in Fig. 133.

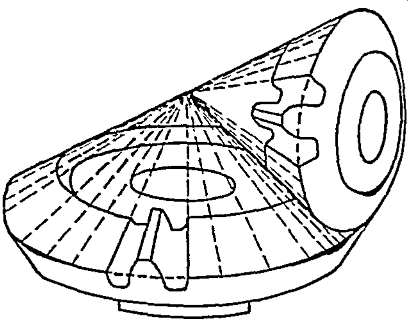


Fig. 133.

The principal distinction between spur-gear teeth and bevel-gear teeth is that in the former the sides of the teeth are parallel and in the latter they are tapered and come to a common point. This point is the intersection of the axes of the two shafts and is called the vertex of the gears.

The following diagram illustrates the various parts of bevel gears:

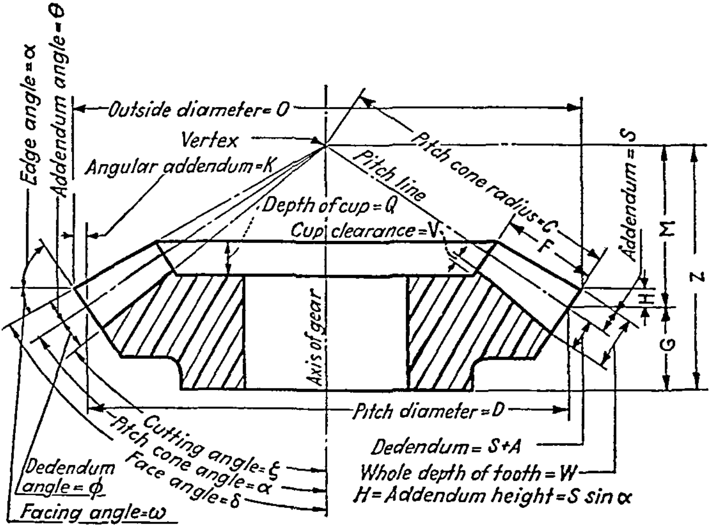


FIG. 134.

NOTATION

P = diametral pitch
N = number of teeth in gear
N' = number of teeth in equivalent spur gear at outer edge
n = number of teeth in pinion
α = pitch-cone angle
γ = shaft angle
D = pitch diameter of gear
d = pitch diameter of pinion
$S + A$ = dedendum
S = addendum
W = whole depth of tooth space
T = thickness of tooth at pitch line
C = pitch-cone radius

F = width of face
s = addendum at small end of tooth
t = thickness of tooth at small end
θ = addendum angle
ϕ = dedendum angle
δ = face angle
ζ = cutting angle
K = angular addendum
ω = facing angle
O = outside diameter
H = addendum height
Z = apex distance
Q = depth of cup
V = cup clearance
M = vertex to pitch diameter
G = hub distance

Formulas for Bevel Gears

No.	To find		Formula
1	Pitch-cone angle of gear		$\tan \alpha_g = \frac{N}{n}$
2	Pitch-cone angle of pinion		$\tan \alpha_p = \frac{n}{N}$
3	Pitch diameter of gear		$D = \frac{N}{P}$
4	Pitch diameter of pinion		$d = \frac{n}{P}$
5	Addendum	These formulas are the same for both gear and pinion	$S = \frac{1}{P}$
6	Dedendum		$S + A = \frac{1.157}{P}$
7	Whole depth of tooth		$W = \frac{2.157}{P}$
8	Pitch-cone radius		$C = \frac{D \csc \alpha}{2}$
9	Addendum at small end		$s = \frac{S(C - F)}{C}$
10	Thickness of tooth on pitch line at small end		$t = \frac{T(C - F)}{C}$
11	Addendum angle		$\tan \theta = \frac{S}{C}$
12	Dedendum angle		$\tan \phi = \frac{S + A}{C}$
13	Number of teeth in equivalent spur gear		$N' = N \sec \alpha$

Formulas for Bevel Gears.—(*Continued*)

No.	To find	Formula
14	Face angle	$\delta = \alpha + \theta$
15	Cutting angle	$\zeta = \alpha - \phi$
16	Angular addendum	$K = S \cos \alpha$
17	Outside diameter	$O = D + 2K$
18	Facing angle	$\omega = 90° - (\alpha + \theta)$
19	Addendum height	$H = S \sin \alpha$
20	Depth of cup	$Q = \left[\frac{W(C-F)}{C} + V\right] \sin \alpha$
21	Vertex to pitch diameter	$M = \frac{N}{2P} \cot \alpha$
22	Apex distance	$Z = M + G$

The **pitch line** is the straight line in the lateral surface of the pitch cone passing through the vertex.

The **pitch-cone angle** is the angle between the axis of the cone and the pitch line. The pitch-cone angle in bevel gears is just as important as the pitch diameter in spur gears.

The **addendum angle** is the angle between the pitch line and the line along the top of the tooth passing through the vertex.

The **dedendum angle** is the angle between the pitch line and the line along the base (or root) of the tooth passing through the vertex.

The **face angle** is the sum of the pitch-cone angle and the addendum angle.

The **root angle** is the pitch-cone angle minus the dedendum angle.

The **pitch-cone radius** is the distance measured on the pitch line from the vertex to the outer edge of the tooth.

The **pitch diameter** is the diameter of the circle formed by the extremities of the pitch-cone radii.

The **addendum** is the length of that portion of the tooth which extends above the pitch line measured on the outer edge of the tooth.

The **dedendum** is the length of that portion of the tooth which falls below the pitch line measured on the outer edge of the tooth.

The addendum and dedendum are both perpendicular to the pitch-cone radius.

The **angular addendum** is the projection of the addendum on the prolongation of the pitch diameter and is equal to the product of the addendum and the cosine of the pitch-cone angle.

The **outside diameter** is equal to the pitch diameter plus two angular addendums.

The **facing angle** is the complement of the face angle and is that angle to which the compound on a lathe is set when cutting bevel gears as in Fig. 135.

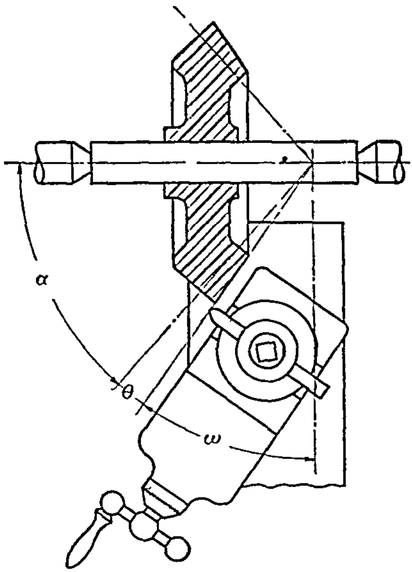


FIG. 135.

The pitch diameter of any bevel gear is perpendicular to its own axis and is equal to the pitch diameter of a spur gear having the same number of teeth and the same diametral pitch.

The pitch-cone angles of a pair of bevel gears whose shaft angle is 90° are included in a right triangle having a base equal to one-half the pitch diameter of one gear and an altitude equal to one-half the pitch diameter of the other gear. The

tangent of either pitch-cone angle is equal to one-half the pitch diameter of the one gear divided by one-half the pitch diameter of the other. It is evident that these half diameters bear the same ratio to each other as do the numbers of teeth in the gears. Consequently, the number of teeth in the gear divided by the number of teeth in the pinion is equal to the tangent of the pitch-cone angle of the gear. Likewise, the number of teeth in the pinion divided by the number of teeth in the gear is equal to the tangent of the pitch-cone angle of the pinion. These relations are expressed in formulas 1 and 2 on page 187.

The **face width** of a bevel-gear tooth is the distance measured along the pitch line from the inner to the outer edge of the tooth. This face width should never exceed one-third of the pitch-cone radius.

The **diametral pitch** and **circular pitch** have the same meaning as in spur gears, the diametral pitch being the number of teeth in π in. of the pitch circumference, while the circular pitch is the distance from the center of one tooth to the center of the next measured on the pitch circumference at the outer edge of the gear.

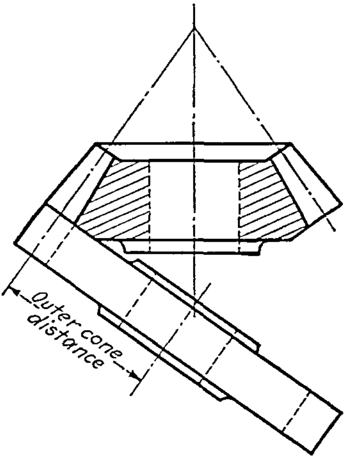


FIG. 136.

The **chordal thickness** and **corrected addendum** of a bevel gear at the large end of the tooth are very nearly equal to (close enough for all practical purposes) the chordal thickness and corrected addendum of a spur gear, having a pitch radius equal to the outer cone distance and having the same diametral pitch. The outer cone distance is the distance along the perpendicular to the pitch-cone radius from the large end of the tooth to where it intersects the axis of the gear, as shown in Fig. 136. The number of teeth in the spur gear at the outer edge of the bevel gear is equal to the product of the number of teeth in the bevel gear and the secant of the pitch-cone angle. After the number of teeth in this spur gear has been determined, the chordal thickness and the corrected addendum may be computed by the aid of the spur-gear chart given on page 159.

When the numerical value of a certain part at the large end of a bevel-gear tooth is known, the corresponding value at the small end can be determined by ratio and proportion as can be seen in Fig. 137.

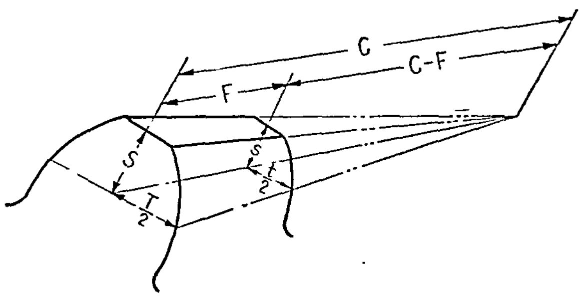


FIG. 137.

Thus:

$$\frac{t}{2}:\frac{T}{2}::C - F:C \qquad \text{or} \qquad t = T\frac{C - F}{C}.$$

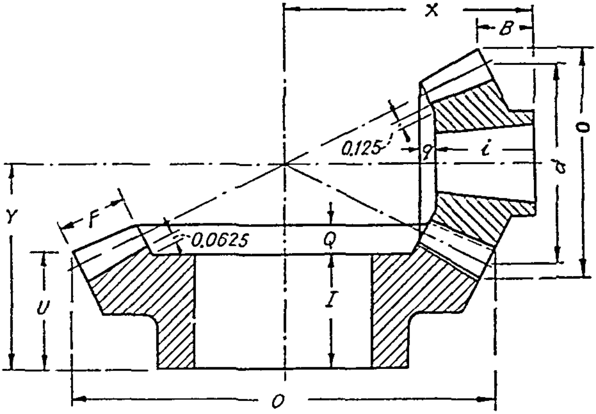
Similarly,

$$s = S\frac{C - F}{C}.$$

Miter gears are a pair of gears each of which has a pitch-cone angle of 45°. Hence both gears have the same number

of teeth and the shaft angle is 90°. The formulas given for bevel gears also apply to miter gears.

PROBLEMS



In the above pair of bevel gears, $N = 45$, n = variable, $P = 6$, $x = 4.5$, $y = 3.25$, and $F = 1.375$.

VARIABLE

1. $n = 22$	2. $n = 24$	3. $n = 26$
4. $n = 28$	5. $n = 30$	6. $n = 32$

Determine:

1. The pitch-cone angle of pinion.
2. The addendum angle.
3. The pitch-cone radius.
4. The pitch diameter of pinion.
5. The distance U.
6. The distance B.
7. The outside diameter of gear.
8. The outside diameter of pinion.
9. The cup depth of gear = Q.
10. The cup depth of pinion = q.

Again using the foregoing figure, consider the given data as follows:

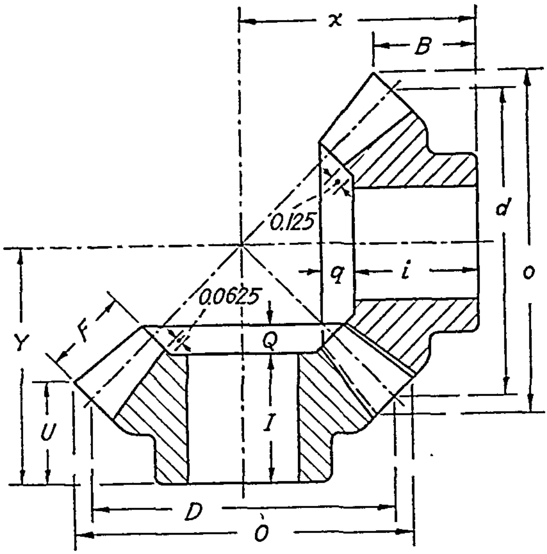
N = variable, $n = 40$, $P = 9$, $x = 3.875$, $y = 3.375$

VARIABLE

1. $N = 54$	2. $N = 55$	3. $N = 56$
4. $N = 57$	5. $N = 58$	6. $N = 59$

Determine:

11. The distance U.
12. The distance B.
13. The distance O.
14. The distance o.
15. The distance I.
16. The distance F.
17. The pitch-cone angle of gear.
18. The whole depth of tooth.
19. The face angle of gear.
20. The cutting angle of pinion.



In the above miter gears, the following data are given:
N = variable, $P = 6$, $x = 4.0625$, $y = 3.9375$.
$F = \frac{1}{3}$ pitch cone radius

VARIABLE

1. $N = 33$	2. $N = 34$	3. $N = 35$
4. $N = 36$	5. $N = 37$	6. $N = 38$

Determine:

21. The distance U.
22. The distance B.
23. The distance d.
24. The distance O.
25. The distance o.
26. The distance I.
27. The distance i.
28. The distance F.
29. The face angle.
30. The cutting angle.

BEVEL GEARS HAVING SHAFT ANGLES LESS THAN OR GREATER THAN 90°

The formulas already developed for bevel gears having a shaft angle of 90° also apply to bevel gears having shaft angles less than or greater than 90°, except formulas 1 and 2 for determining pitch-cone angles of gear and pinion. Special formulas will now be developed for these pitch-cone angles.

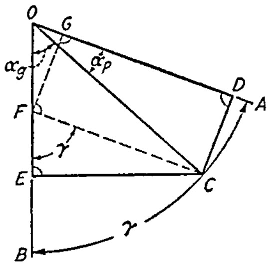


FIG. 138.

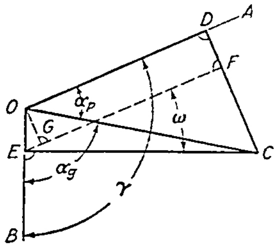


FIG. 139.

To prove that cotangent of the pitch-cone angle of the gear is:

$$\cot \alpha_g = \frac{n \csc \gamma}{N} + \cot \gamma.$$

Proof: CE is one-half the pitch diameter of the gear and CD is one-half the pitch diameter of the pinion. In Fig. 138, let CE, which is equal to $\frac{N}{2P}$, be unity. If all parts of the figure are reduced on the same scale, CD, which was $\frac{n}{2P}$, becomes $\frac{n}{N}$ and EF becomes $\cot \gamma$. Draw CF parallel to AO and FG parallel to CD. Then $FG = CD = \frac{n}{N}$. On the same reduction scale, OF becomes $\frac{n}{N} \csc \gamma$.

$$OE = OF + FE = \frac{n \csc \gamma}{N} + \cot \gamma.$$

Since CE is unity, $\quad OE = \cot \alpha_g$.

Hence $\quad \cot \alpha_g = \frac{n}{N} \csc \gamma + \cot \gamma.$

Similarly it may be proved that

$$\cot \alpha_p = \frac{N}{n} \csc \gamma + \cot \gamma.$$

For shaft angles greater than 90°, the student should derive the following formulas with the aid of Fig. 139.

$$\cot \alpha_g = \frac{n}{N} \sec \omega - \tan \omega.$$

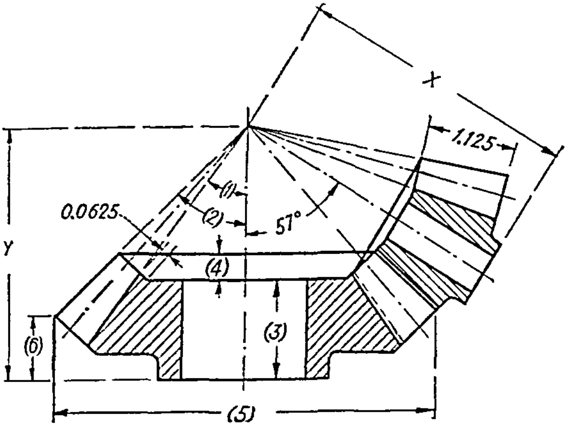
and

$$\cot \alpha_p = \frac{N}{n} \sec \omega - \tan \omega.$$

where

$$\omega = \gamma - 90°.$$

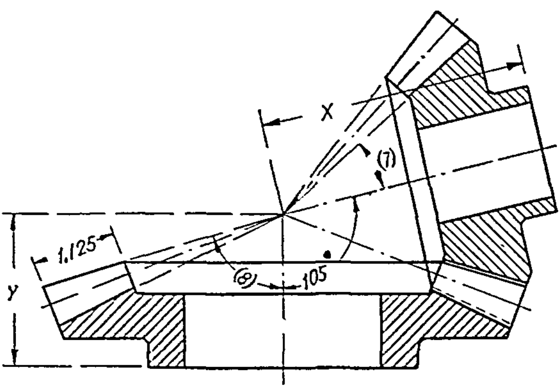
PROBLEMS



Variable		
No.	Sym.	Value
1	N	29
2	N	30
3	N	31
4	N	32
5	N	33
6	N	34

In the above pair of bevel gears, the following data are given: N = variable, $n = 17$, $P = 4$, $x = 6.875$, $y = 6.9375$.

1.–6. Determine the angles and distances indicated by the numbers in parentheses on the foregoing diagram.



Variable		
No.	Sym.	Value
1	N	36
2	N	38
3	N	40
4	N	35
5	N	37
6	N	39

In the above pair of bevel gears, the following data are given: N = variable, $n = 17$, $P = 5$, $x = 4.250$, $y = 2.3125$.

7. and **8.** Determine the angles indicated by (7) and (8) of the foregoing figure.

CHECKING THE FACE DISTANCES IN BEVEL-GEAR HOUSINGS BY MEANS OF PLUG GAGES—SHAFT ANGLES LESS THAN OR GREATER THAN 90°

Checking for the thickness of the collar that is fitted between the gear housing and the hub face of the gear is usually done if possible by feeling with the finger to make sure that the outer cone surface of the two gears are flush and by then checking the distance between the faces of the gear and the housing with a thickness gage which varies in thousandths of an inch.

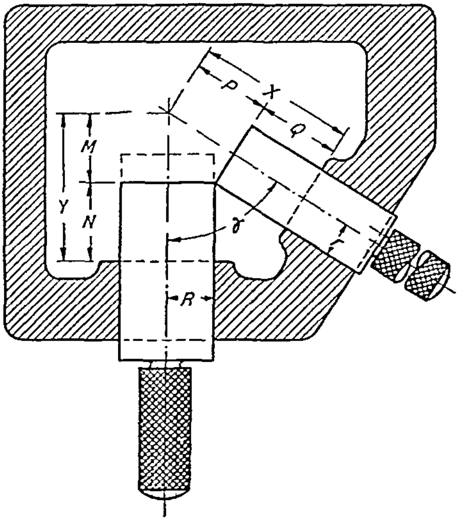


FIG. 140.

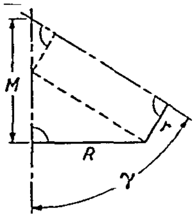


FIG. 141.

However, in most cases when the gears are in the housing to be fitted, there is not sufficient room to enable one to feel the outer cone surfaces of the gears. A better and more accurate method is to machine the faces in the gear housing to their proper dimensions x and y as shown in Figs. 140 and 142. This is accomplished by the aid of a depth micrometer and plug gages which are used to check the distances x and y in the gear housing until the proper dimensions have been attained. The top surfaces of the plug gages must be perpendicular to their axes.

If the distance from the outside diameter of the gear to the hub face, measured parallel to the axis, is correctly machined and if the distances x and y in the gear housing are held within close limits, then the above method will necessitate very little fitting and the pair of bevel gears will function at their highest efficiency.

For shaft angles less than 90° the formulas to be used are as follows, using the notation of Figs. 140 and 141.

$$M = R \cot \gamma + r \csc \gamma.$$
$$P = r \cot \gamma + R \csc \gamma.$$
$$N = Y - M.$$
$$Q = X - P.$$

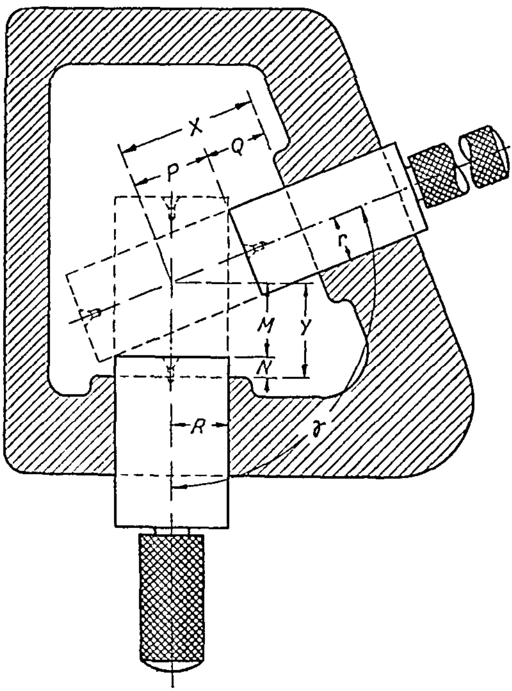


Fig. 142.

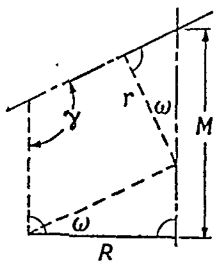


Fig. 143.

For shaft angles greater than 90° the formulas to be used are as follows, using the notation of Figs. 142 and 143.

$$M = r \sec \omega + R \tan \omega.$$
$$P = R \sec \omega + r \tan \omega.$$

where

$$\omega = \gamma - 90°.$$
$$N = Y - M.$$
$$Q = X - P.$$

PROBLEMS

Referring to Fig. 140.

$$x = 6.875,\ y = 6.9375,\ r = .875, \text{ and } R = 1.0625.$$

VARIABLE

1. $\gamma = 55°$	**2.** $\gamma = 57°$	**3.** $\gamma = 59°$
4. $\gamma = 61°$	**5.** $\gamma = 69°$	**6.** $\gamma = 81°$

1. Determine the distance N. **2.** Determine the distance Q.
Referring to Fig. 142.

$$x = 4.25,\ y = 2.3125,\ r = .5625, \text{ and } R = .6875.$$

VARIABLE

1. $\gamma = 105°$	**2.** $\gamma = 98°$	**3.** $\gamma = 115°$
4. $\gamma = 110°$	**5.** $\gamma = 108°$	**6.** $\gamma = 121°$

3. Determine the distance N. **4.** Determine the distance Q.

BEVEL-GEAR ANCHORS

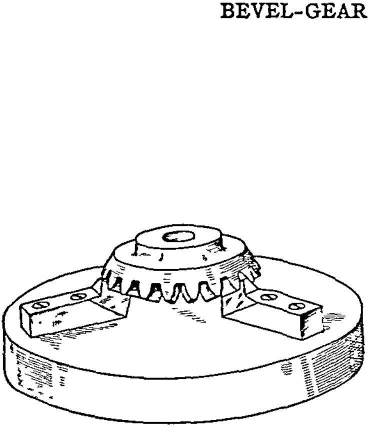


FIG. 144.

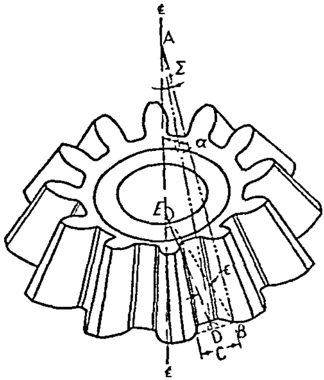


FIG. 145.

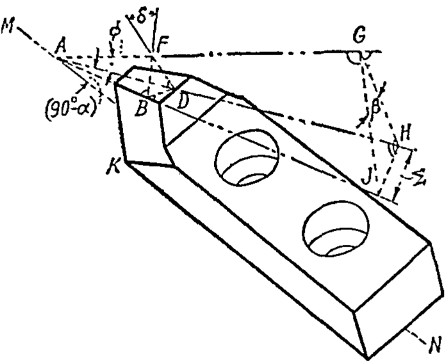


FIG. 146.

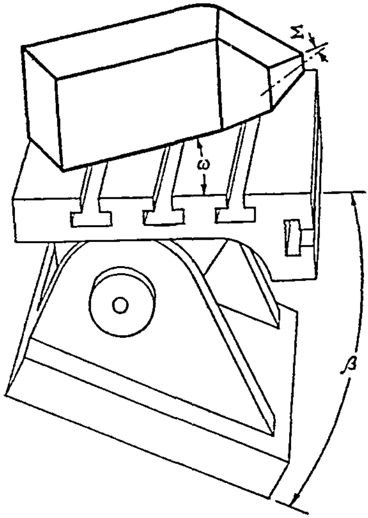


FIG. 147.

In order that the hub face and bore of a hardened bevel gear may be ground concentric with the tooth formation, it is necessary to mount the gear on three anchors as shown in Fig. 144. The dimensions and angles of these anchors must be determined from the data given for the bevel gears. Figure 146 shows the angle that must be determined for one of these anchors in order that it may be ground on a fixture as shown in Fig. 147. The angles in Fig. 146 depend on the number of teeth and the pitch-cone angle of the bevel gear. Their relations are shown in Fig. 145. AB is the pitch-cone radius, EB is the pitch radius, angle EAB is the pitch-cone angle α, BD is one-half the chordal thickness of a space measured on the pitch circumference, and the pressure angle of the gear is δ. The angle DEB, or ϵ, lies in the plane of the pitch circumference and is equal to 90° divided by the number of teeth. If these two angles α and ϵ are known, the angle Σ (which is shown in both Figs. 145 and 146) may be determined by the use of the compound-angle Type I figure AED-B.

The plane of the pressure angle of a bevel gear is perpendicular to the pitch-cone radius lying in the surface of a tooth, which is line AB in both Figs. 145 and 146. Thus the plane

WORM GEARS

Worm gears are used to transmit power from one rotating shaft to another when the shafts do not intersect and are not parallel. The shafts are usually at right angles to each other. There are two distinct advantages in worm gearing: first, worm gearing can be designed greatly to reduce the velocity, thereby increasing the force applied; second, the effective action is smoother than that of any other type of gears.

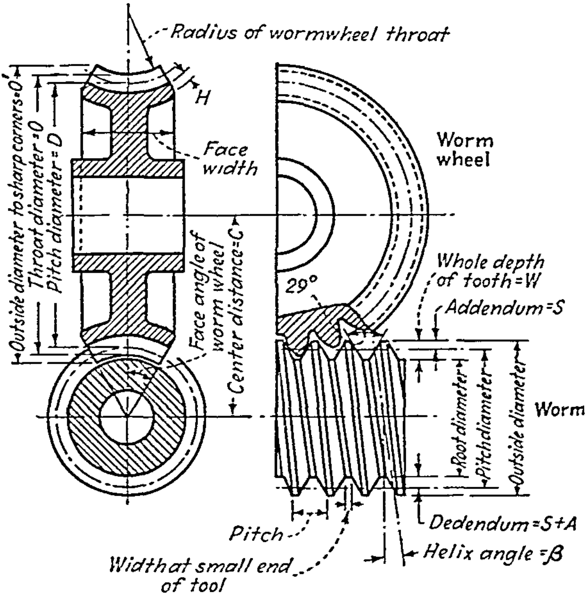


FIG. 151.

The two units of this type of gearing are known as the worm, which very closely resembles a thread screw, and the worm wheel, which if cut perpendicular to the axis through the lowest point of the throat radius resembles a spur gear having angular teeth. The mechanical action of spur-gear teeth is a rolling action while that of a worm and the worm wheel is a combination of rolling and sliding. The worm is made of low carbon steel and is cynanided or case-hardened, while the worm wheel is made of bronze. Worm and worm wheels

should always run in oil or light grease, in which case the efficiency is nearly as high as that of spur gears.

NOTATION

P = circular pitch of worm wheel and linear pitch of worm
L = lead of worm
n = number of threads wound around worm
S = addendum, or height of worm tooth above the pitch line
d = pitch diameter of worm
D = pitch diameter of worm wheel
o = outside diameter of worm
O = throat diameter of worm wheel
N = number of teeth in worm wheel
W = whole depth of tooth
F = width of thread tool at end
ϕ = face angle of worm wheel
β = helix angle of worm and gashing angle of worm-wheel tooth
U = radius of curvature of worm-wheel throat
C = center distance
M = minimum length of worm
O' = outside diameter of worm wheel to corners
A = clearance
T = chordal thickness of worm wheel
t = thickness of worm tooth at pitch line
Q = width of worm-wheel face
H = distance from root of tooth to the face measuring along beveled edge of worm wheel

The **linear pitch** is the distance from the center of one thread to the center of the next, measured along a line parallel to the axis. This pitch is usually expressed as a common fraction.

The expression **number of threads** means the number of threads wound around the worm and not the number of threads per inch. It is generally spoken of as the number of starts. For example, a double-threaded worm is referred to as two starts, a triple-threaded worm as three starts, etc.

The **lead** is the distance a thread advances, along a line parallel to the axis, in one complete revolution. The lead is equal to the product of the linear pitch and the number of threads on a multiple-threaded worm. Hence, the linear pitch on a multiple-threaded worm is equal to the lead divided by the number of threads. The student should thoroughly

understand the distinction between the lead and the linear pitch.

Formulas for Worm Gears

No.	To find	Formulas
1	Linear pitch	$P = \frac{L}{n}$
2	Lead	$L = n \times P$
3	Addendum	$S = .3183P$
4	Pitch diameter of worm wheel	$D = .3183NP$
5	Pitch diameter of worm	$d = 2C - D$
6	Center distance	$C = \frac{D + d}{2}$
7	Whole depth of tooth	$W = .6866P$
8	Helix angle of worm	$\cot \beta = \frac{3.1416d}{L}$
9	Width of thread tool at small end	$F = .31P$
10	Throat diameter of worm wheel	$O = D + 2S$
11	Radius of worm-wheel throat	$U = C - \frac{O}{2}$
12	Diameter of worm-wheel sharp corners	$O' = 2[C - U \cos \phi]$
13	Minimum length of worm	$A' = 2\sqrt{2S(D - 2S)}$
14	Pitch diameter of worm	$d = o - 2S$
15	Chordal thickness of worm-wheel tooth	$T = D \sin \frac{90°}{N} \cos \beta$
16	Thickness of worm tooth at pitch line	$t = \frac{P}{2}$
17	Clearance	$A = .05P$
18	Width of worm-wheel face	$Q = (o + .1P + 2H) \sin \phi$

The **helix angle** of a worm is the angle between the helix of the thread and the plane perpendicular to the axis of the worm at the pitch diameter. The mechanical efficiency of a worm is highest when the helix angle is numerically small.

The **throat** is the curved surface located at the extremities of the minimum outside diameter, called the throat diameter, of the worm wheel. The cross section through the axis of the worm wheel of this curved surface is an arc of a circle the radius of which is called the throat radius and is equal to one-half the outside diameter of the worm minus two addendums.

Do not confuse this dimension with one-half the throat diameter of the worm wheel.

When the helix angle of a worm is greater than 8°, the dimensions of the thread and tooth should be measured at right angles to the thread and tooth, respectively.

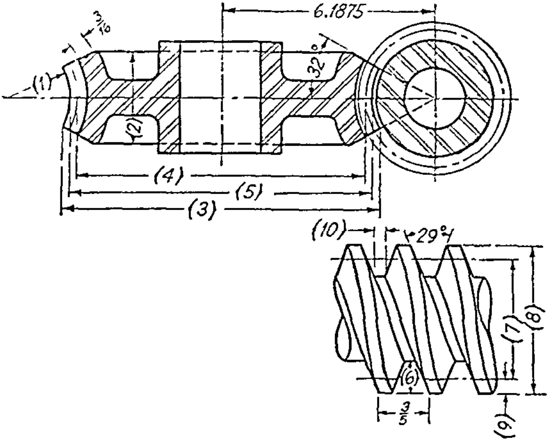
The **normal circular pitch** (P_n) is equal to the product of the linear pitch and the cosine of the helix angle. For helix angles greater than 8°, use the normal circular pitch instead of the linear pitch for computing the addendum, whole depth of tooth, thickness of tooth, and width of thread tooth at small end. The formulas for these quantities become:

3′. $S = .3183P_n$, 7′. $W = .6866P_n$, 9′. $F = .31P_n$,

16′. $t = \frac{P_n}{2}$.

The width of the worm gear face should never exceed two-thirds of the outside diameter of the worm. The face angle ϕ of the worm wheel (Fig. 151) is usually from 30° to 35°. As this angle increases, the width of the worm-wheel face increases accordingly, causing the worm wheel to wear out more rapidly, on account of the double slippage action at the corner of the tooth. Owing to this slippage, the outside diameter to sharp corners (O' of Fig. 151) is usually somewhat less than that given by the formula.

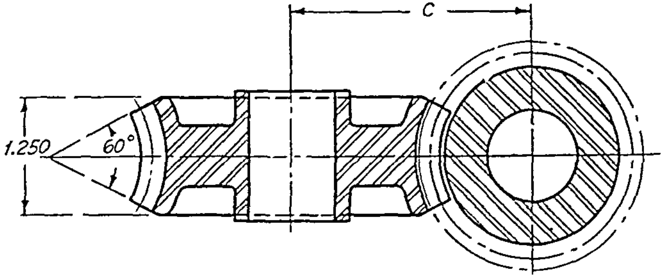
PROBLEMS



VARIABLE		
No.	Sym.	Value
1	N	44
2	N	45
3	N	46
4	N	47
5	N	48
6	N	49

1.–10. From the data given in the foregoing figure and having the variable equal to the number of teeth N, determine the 10 parts of the single-threaded worm and worm wheel indicated by the numbers in the diagram.

11. Determine the helix angle of the worm.

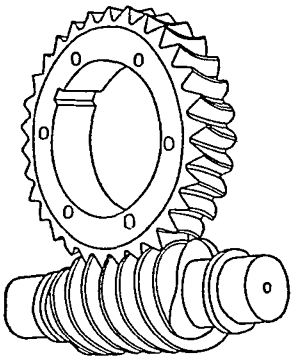


In the single-threaded worm of the above diagram, the following data are given: pitch = .3125, face angle = 30°, outside diameter of worm = 2.1989, N = variable = number of teeth in worm wheel.

VARIABLE		
1. $N = 61$	**2.** $N = 62$	**3.** $N = 63$
4. $N = 64$	**5.** $N = 65$	**6.** $N = 66$

12. Determine the pitch diameter of the worm wheel.

13. Determine the center distance.
14. Determine the throat radius.
15. Determine the outside diameter to sharp corners.
16. Determine the helix angle of the worm.



In the double-threaded worm shown above, the given data are as follows: Pitch = .375, center distance = 4.709, N = variable = number of teeth in worm wheel.

VARIABLE

1. $N = 54$	2. $N = 55$	3. $N = 56$
4. $N = 57$	5. $N = 58$	6. $N = 59$

17. Determine the throat diameter of the worm wheel.
18. Determine the outside diameter of the worm.
19. Determine the helix angle of the worm.

Using the foregoing figure, but considering the worm to be triple-threaded, the pitch = .4375, the face angle = 31°, the center distance = 4.375, and N = variable = number of teeth in worm wheel.

VARIABLE

1. $N = 41$	2. $N = 42$	3. $N = 43$
4. $N = 44$	5. $N = 45$	6. $N = 46$

20. Determine the radius of worm-wheel throat.
21. Determine the outside diameter to sharp corners of worm wheel.
22. Determine the helix angle of the worm.

SPIRAL GEARS

Definitions

Helical gears (commonly known as **spiral gears**) are used to transmit power from one rotating shaft to another, when the shafts do not intersect. They must be made as nearly perfect as possible and should work in an oil bath, as the contact

between the teeth is a sliding action and not a rolling action as in spur gears.

The **normal diametral pitch** is the number of teeth in π in. measured on a line perpendicular to the teeth.

The **normal circular pitch** is the shortest distance from the center of one tooth to the center of the next, measured along a line on the pitch cylinder at right angles to the teeth.

The **real pitch** is the distance from the center of one tooth to the center of the next, measured along the pitch circle. The real pitch can be determined by multiplying the normal circular pitch by the secant of the helix angle.

The **helix angle** is that angle made by a line parallel to the tooth and a line on the pitch cylinder parallel to the axis. The helix angle is the first thing to be determined in the construction of a pair of spiral gears, whose known parts are the center distance, ratio between the gears, and the normal pitch (diametral or circular).

The **lead** is the distance a tooth advances along a line parallel to the axis in one complete revolution.

The **normal chordal thickness**, which is measured at the pitch line, is equal to the chordal thickness of a spur gear having the same number of teeth and the same pitch and may be determined from the spur-gear chart accompanying Fig. 119 on page 159.

The **corrected addendum** of a spiral gear is equal to the corrected addendum of a spur gear whose pitch diameter is equal to the pitch diameter of the spiral gear and whose number of teeth is equal to the product of the pitch diameter of the spiral gear and the diametral pitch. When the number of teeth is thus obtained, the spur-gear chart accompanying Fig. 119 on page 159 is used to get the corrected addendum.

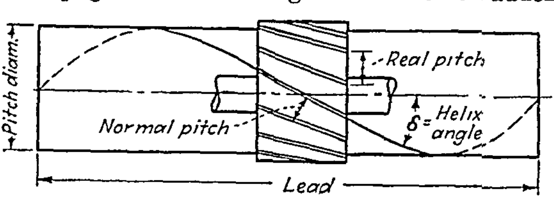


FIG. 152.

In Fig. 152, the essential parts of a spiral gear are shown in their relative positions. Notice that increasing the helix angle increases the real pitch, thereby increasing the pitch circumference (for a given number of teeth). From this statement it is evident that there are three things which govern the pitch diameter of a pair of spiral gears, *viz.*, the number of teeth, the normal pitch, and either the center distance or the helix angle. In order to solve any spiral-gear problem, the foregoing three parts must be given.

Spiral gears are the only gears in which the pitch diameters are independent of the number of teeth, the latter merely specifying the ratio required. Thus two gears of the same diametral pitch, and having the same pitch diameter, may have 25 and 35 teeth. In cases like this the gear having the lesser number of teeth would have the greater helix angle.

CORRESPONDING SPUR GEARS

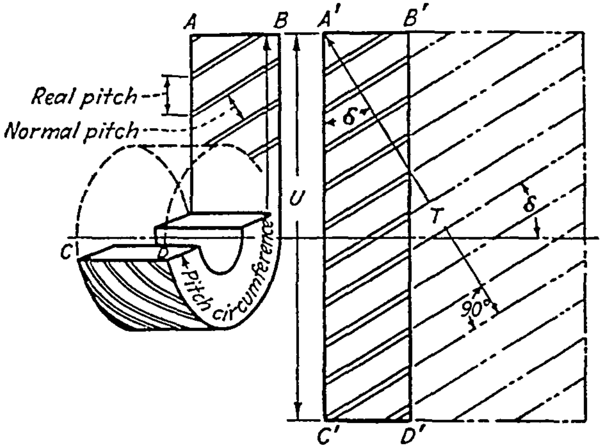


FIG. 153.

NOTATION

U = pitch circumference of spiral gear
T = pitch circumference of corresponding spur gear
δ = helix angle

Every spiral gear, whether the shafts are parallel or angular, has a corresponding spur gear. The pitch circumference of the corresponding spur gear is always perpendicular to the teeth. Figure 153 represents a spiral gear which has been

turned down to the pitch diameter. If this gear were rolled one complete revolution on a piece of paper, it would produce the impression marks as shown on the diagram $A'B'C'D'$. These impression marks merely represent the development of the spiral gear teeth. U represents the pitch circumference of the spiral gear and T the pitch circumference of the corresponding spur gear of the same diametral pitch. It is evident that these pitch circumferences bear the same ratio to each other as do their diameters. Therefore, the pitch diameter of the corresponding spur gear divided by the pitch diameter of the spiral gear is equal to the cosine of the helix angle. That is, the product of the pitch diameter of the spur gear and the secant of the helix angle is equal to the pitch diameter of the spiral gear.

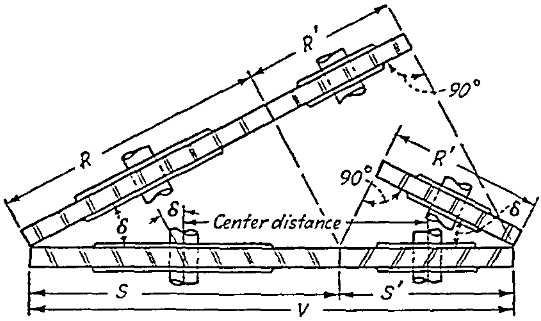


Fig. 154.

Notation

S = pitch diameter of spiral gear
S' = pitch diameter of spiral pinion
R = pitch diameter of corresponding spur gear
R' = pitch diameter of corresponding spur pinion
V = twice center distance
δ = helix angle.

Figure 154 shows the relation between the spiral gear and the corresponding spur gear, when the shafts are parallel. From this diagram it may be seen that the cosine of the helix angle is equal to twice the center distance of the corresponding spur gears divided by twice the center distance of the spiral gears.

In order to avoid excessive end thrust in a set of spiral gears whose shafts are parallel, the helix angle should not exceed 20°. To get the greatest efficiency in spiral gears whose shafts are angular, the helix angle should be one-half the shaft angle.

When a spiral gear is placed in a horizontal position, if the direction of the tooth from top to bottom is clockwise, the spiral gear is said to be a right-hand helix; but if the direction of the tooth from top to bottom is counterclockwise, the spiral gear is said to be a left-hand helix.

In duplicating a pair of spiral gears, the following relations are important:

1. *When helixes are of like hands, the shaft angle is the sum of the helix angles.*
2. *When helixes are of unlike hands, the shaft angle is the difference of the helix angles.*
3. *When the shafts are at right angles, the helixes of both gears must be of like hands.*
4. *When the shafts are parallel, the helixes of both gears must be of unlike hands.*
5. *When the shafts are at an acute angle, the helixes of both gears may be either of like or of unlike hands.*

SPIRAL-GEAR NOTATION AND FORMULAS

The following are the general formulas which are to be used when the number of teeth, diametral pitch, helix angle, and the center distance are known. In order to duplicate a set of spiral gears whose known parts are the number of teeth, the center distance, and the shaft angle, first determine the diametral pitch, next the helix angle, then use the formulas below in the order given.

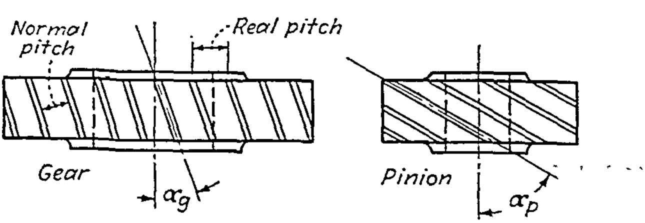


FIG. 155.

NOTATION

P_n = normal diametral pitch
d = pitch diameter of pinion
n = number of teeth in pinion
α_p = helix angle of pinion
C = center distance
S = addendum
W = whole depth of tooth
D = pitch diameter of gear
N = number of teeth in gear
α_g = helix angle of gear
γ = angle between shafts
L = lead of tooth spiral
N' = number of teeth (to select cutter)
O = outside diameter of gear
N_n = number of teeth for determining the corrected addendum by the spur-gear chart of page 159

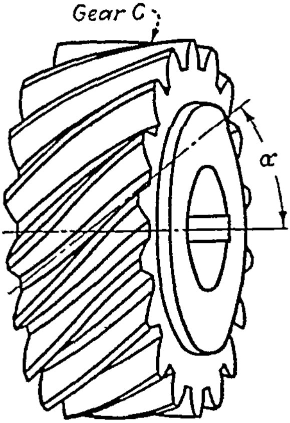
Formulas for Spiral Gearing

No.	To find	Formula
1	Pitch diameter of gear	$D = \frac{N \sec \alpha_g}{P_n}$
2	Pitch diameter of pinion	$d = \frac{n \sec \alpha_p}{P_n}$
3	Lead of gear	$L = \pi D \cot \alpha_g$
4	Lead of pinion	$L = \pi d \cot \alpha_p$
5	Outside diameter of gear	$O = D + 2S$
6	Addendum	$S = \frac{1}{P_n}$
7	Whole depth of tooth	$W = \frac{2.157}{P_n}$
8	Number of teeth for which to select cutter	$N' = \frac{N}{\cos^3 \alpha}$
9	Number of teeth for determining the corrected addendum	$N_n = DP_n$ $N_n = N \sec \alpha$

The following formulas are to be used *only* when the shafts are parallel.

10	Pitch diameter of gear	$D = \frac{2CN}{N + n}$
11	Pitch diameter of pinion	$d = \frac{2Cn}{N + n}$
12	Cosine of the helix angle	$\cos \alpha = \frac{N + n}{2P_nC}$

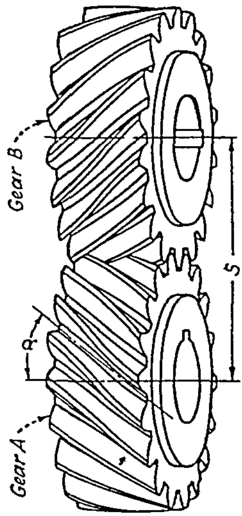
PROBLEMS



VARIABLE		
No.	Sym.	Value
1	T	7
2	T	9
3	T	12
4	T	5
5	T	4
6	T	6

Gear C has 28 teeth, a diametral pitch of T, and a helix angle of 15° 37′.
Determine:

1. The pitch diameter.
2. The outside diameter.
3. The number of cutter.
4. The lead of helix.
5. The addendum.
6. The depth of tooth.
7. The normal chordal thickness.
8. The corrected addendum.
9. The real pitch.



VARIABLE		
No.	Sym.	Value
1	S	3.3951
2	S	3.4072
3	S	3.4353
4	S	3.4504
5	S	3.5054
6	S	3.5256

Gear A has 19 teeth and gear B has 21 teeth. Diametral pitch = 6. S = variable = center distance.
Determine:

10. The pitch diameter of gear A.
11. The pitch diameter of gear B.
12. The helix angle of both gears.
13. The outside diameter of gear B.
14. The outside diameter of gear A.

PROOF OF HELIX-ANGLE FORMULA

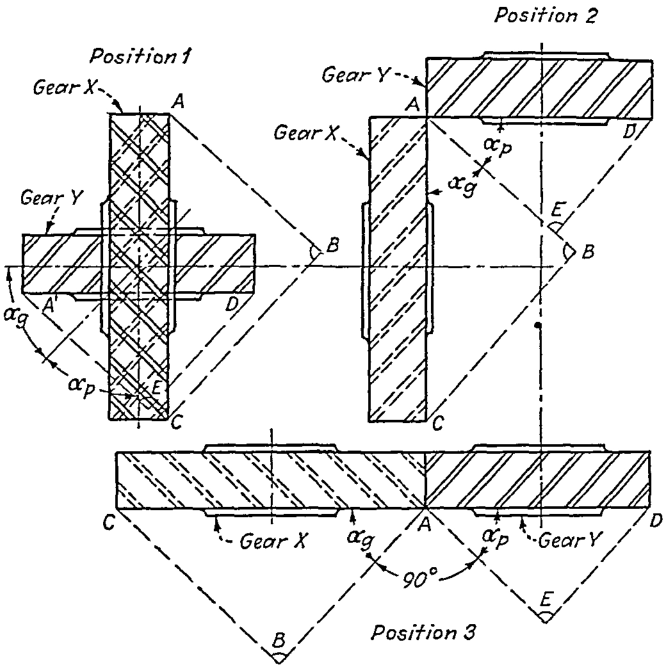


FIG. 156.

Figure 156 shows a set of spiral gears at a 90° shaft angle. The gears are placed in the positions 1, 2, and 3 for the purpose of proving the following formula:

$$R \sec \alpha_g + \sec \alpha_p = \frac{2CP_n}{n}.$$

Position 1 is the original position of the two gears X and Y when engaged. Consider that the teeth of these gears have been cut down to the pitch cylinder. The distances AC and $A'D$ represent the pitch diameters of the two spiral gears X and Y, respectively. The distances AB and $A'E$ represent the pitch diameters of the corresponding spur gears, respectively. Each spiral gear has a corresponding spur gear, and when the pitch diameter of the corresponding spur gear is spoken of, it must be understood that it is perpendicular to the teeth of the spiral gear in the plane of contact.

Position 2 is attained by raising the Y gear until the lower face is at a level with the top edge of the X gear, next bringing it forward until the two center lines intersect, then moving it to the right until point A' coincides with point A. It is evident that the distances AB and AE still represent the pitch diameters of the corresponding spur gears. The helix angle of a spiral gear is that angle made by the tooth and a line parallel to the axis. Therefore, it is evident that the angles CAB and EAD are equal to the helix angles of the X and Y gears, respectively; and since the shafts are at right angles, the sum of the helix angles must be 90°.

Position 3 is attained by rotating the X gear until it is in a straight line with the Y gear. This gear is thus rotated through an angle of 90°, which in turn causes angle BAE to equal 90°. Let C represent the center distance. Then the distance $CD = 2C$. The distances AB and AE are equal to the pitch diameters of the corresponding spur gears. The helix angle of the X gear is equal to the shaft angle minus the helix angle of the Y gear. Using the symbols already given:

$$\overline{AE} = \frac{n}{P_n}. \qquad \overline{AB} = \frac{N}{P_n}. \qquad \overline{CD} = 2C.$$

To reduce AE to unity, divide by $\frac{n}{P_n}$ or multiply by $\frac{P_n}{n}$. In order to keep the diagram in the same shape, it is necessary to multiply all other quantities by this same fraction.

Therefore

$$AB = \frac{N}{P_n} \times \frac{P_n}{n} = \frac{N}{n} \qquad \text{and} \qquad CD = 2C \times \frac{P_n}{n}.$$

Since AE is now unity, the distance AD becomes equal to sec α_p, and since $AB = \frac{N}{n}$ the distance AC becomes $\frac{N}{n}$ sec α_g. Let $R = \frac{N}{n}$. Then, since $AC + AD = CD$, we have

$$R \sec \alpha_g + \sec \alpha_p = \frac{2CP_n}{n}.$$

Solving the foregoing equation for the number of teeth in the pinion:

$$n = \frac{2CP_n}{R \sec \alpha_g + \sec \alpha_p}.$$

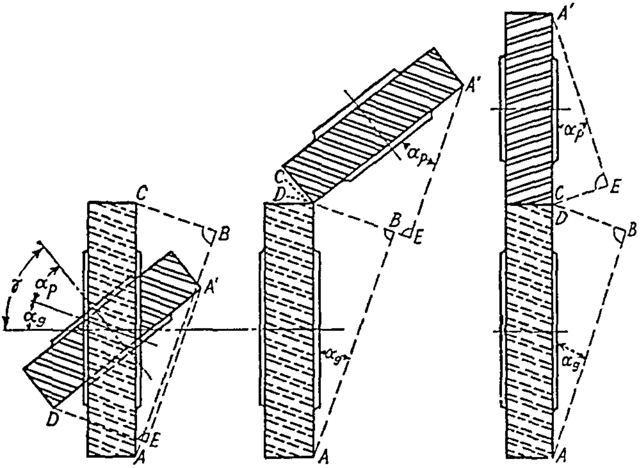


FIG. 157.

In the foregoing diagram the corresponding three positions for two left-hand spiral gears having an oblique shaft angle

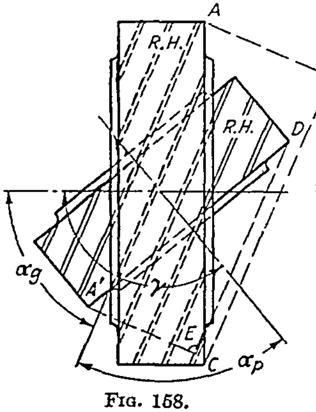


FIG. 158.

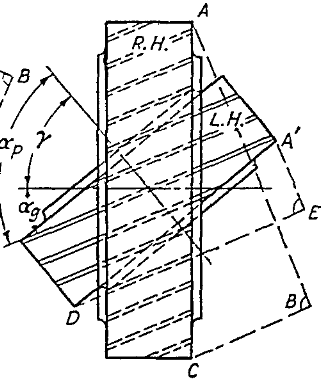


FIG. 159.

are shown, and the student is asked to develop the helix-angle formula which will be found to be the same as when the shaft angle is 90°.

Figures 158 and 159 show, in their original positions, two sets of spiral gears having oblique shaft angles and of like hands and unlike hands, respectively. For each of these cases the student should draw diagrams showing the other two positions and should develop the helix-angle formula which again will be the same as in the original case where the shaft angle was 90°.

DESIGNING A SET OF SPIRAL GEARS

The data usually given in designing a set of spiral gears are the diametral pitch, the center distance, and the ratios of the numbers of teeth in the two gears. The helix angle is the first quantity to be determined. There are two methods of computing this helix angle: the algebraic, which is quite complicated, and the cut and try, which is most generally used. In duplicating a set of spiral gears the number of teeth in the pinion can be counted, but in designing a new set of spiral gears the number of teeth must first be computed by using the formula already derived which is

$$n = \frac{2CP_n}{R \sec \alpha_g + \sec \alpha_p}.$$

Illustrative Example:

Given: Shaft angle = 90°, diametral pitch = 8, center distance = 5.5 inches, ratio of numbers of teeth in gears = 3 to 2. The gear is to be placed on the driving shaft and the pinion on the driven shaft. (The helix angle of a driving gear always is larger than the helix angle of the driven gear. Either the gear or the pinion may be the driving gear.)

Solution: Since the efficiency of the spiral gears is highest when the helix angle is equal to one-half the shaft angle, it is advisable to start with the assumption that both helix angles are 45°. Applying the formula for getting the number of teeth in the pinion:

$$n = \frac{2 \times 5.5 \times 8}{1.5 \sec 45° + \sec 45°} = \frac{88}{3.5355} = 24.9.$$

This indicates that the number of teeth in the pinion must be either 24 or 25. The correct one of these numbers is the one which will give a helix angle of the driving gear greater than the helix angle of the driven gear. This must be found by trial.

If it is assumed that the pinion has 25 teeth, the right-hand member of the helix-angle equation

$$\left(R \sec \alpha_g + \sec \alpha_p = \frac{2CP_n}{n}\right)$$

can be evaluated. $\dfrac{2CP_n}{n} = \dfrac{2 \times 5.5 \times 8}{25} = 3.5200.$

Thus,

$$R \sec \alpha_g + \sec \alpha_p = 3.5200.$$

For future reference, the numerical value of the right-hand member will be called the quotient.

Using α_g and α_p both equal to 45° gives

$$1.5 \sec 45° + \sec 45° = 3.5355.$$

This value is larger than the quotient, which means that the values of α_g and α_p must be different. Since α_g must be greater than α_p and since their sum must equal 90°, try α_g greater than 45°, say 47°; in this case $\alpha_p = 43°$.

Then

$$1.5 \sec 47° + \sec 43° = 3.5667.$$

The fact that this value is still farther from the quotient shows that the wrong value must have been chosen for the number of teeth in the pinion.

Using $n = 24$ gives $\dfrac{2CP_n}{n} = \dfrac{2 \times 5.5 \times 8}{24} = 3.6666$ for the new quotient.

Continue to substitute degrees in the left-hand member of the helix equation until two sets of degrees have been obtained which give two results, one of which is greater than and the other less than the value of the computed quotient. The two helix angles of the same gear of these two final sets must differ by only 1°.

In this problem the work is as follows:

$$1.5 \sec 45° + \sec 45° = 3.5355.$$
$$1.5 \sec 47° + \sec 43° = 3.5667.$$
$$1.5 \sec 49° + \sec 41° = 3.6113.$$
$$1.5 \sec 50° + \sec 40° = 3.6389.$$
$$1.5 \sec 51° + \sec 39° = 3.6702.$$

The final two steps give two results, one of which is greater than and one of which is less than the quotient.

Use the general interpolation method to determine the approximate number of minutes corresponding to the quotient. Thus:

$$\begin{array}{rl} 3.6666 &= \text{quotient} \\ 3.6389 &= \text{lesser number} \\ \hline 277 &= \text{difference} \end{array} \qquad \begin{array}{rl} 3.6702 &= \text{greater number} \\ 3.6389 &= \text{lesser number} \\ \hline 313 &= \text{difference} \end{array}$$

$$\tfrac{277}{313} \times 60 = 53'.$$

Checking this probable value of 50° 53′ for α_g and $90° - 50°\ 53' = 39°\ 7'$ for α_p:

$$1.5 \sec 50°\ 53' + \sec 39°\ 7' = 3.6664.$$
$$1.5 \sec 50°\ 54' + \sec 39°\ 6' = 3.6670.$$

Similarly, interpolating to get seconds:

$$\begin{array}{rl} 3.6666 &= \text{quotient} \\ 3.6664 &= \text{lesser number} \\ \hline 2 &= \text{difference} \end{array} \qquad \begin{array}{rl} 3.6670 &= \text{greater number} \\ 3.6664 &= \text{lesser number} \\ \hline 6 &= \text{difference} \end{array}$$

$$\tfrac{2}{6} \times 60 = 20''.$$

Checking this probable value of 50° 53′ 20″ for α_g:

$$1.5 \sec 50°\ 53'\ 20'' + \sec 39°\ 6'\ 40'' = 3.6666.$$

Hence the helix angle of the gear is 50° 53′ 20″ and the helix angle of the pinion is 39° 6′ 40″.

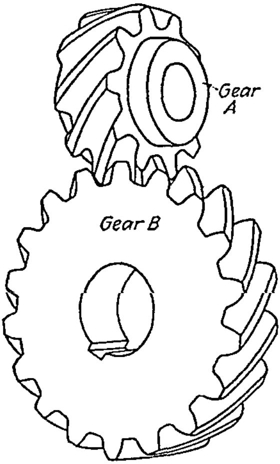
Check these helix angles by computing the center distance.

$$C = \frac{N \sec \alpha_g + n \sec \alpha_p}{2P_n} = \frac{36 \times 1.5852 + 24 \times 1.2887}{2 \times 8}$$
$$= 5.49975.$$

This slight variation from the given center distance 5.5 indicates that no appreciable error has been made.

Note: Sometimes, in order to make the helix angle of the driving gear greater than the helix angle of the driven gear, varying only the value of n produces helix angles far from half the shaft angle. In this case it is best to alter either the diametral pitch or the center distance.

PROBLEMS



For the following ten problems, refer to the above figure.

Data for Problems 1 and 2:
Center distance $= U =$ variable
Diametral pitch $= 8$
Number of teeth in gear $= 40$
Number of teeth in pinion $= 18$
Shaft angle $= 90°$

Variable

1. $U = 5.0807$	**2.** $U = 5.0763$	**3.** $U = 5.1100$
4. $U = 5.0405$	**5.** $U = 5.0504$	**6.** $U = 5.0176$

1. Determine the helix angle of pinion A.

2. Determine the pitch diameter of gear B.

Data for Problems 3 and 4:
Center distance $= T =$ variable
Diametral pitch $= 20$

Number of teeth in gear = 90
Number of teeth in pinion = 30
Shaft angle = 90°

VARIABLE

1. $T = 4.1422$	2. $T = 4.2245$	3. $T = 4.1862$
4. $T = 4.1913$	5. $T = 4.1103$	6. $T = 4.1550$

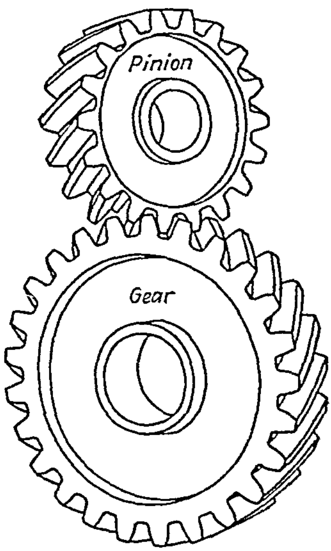
3. Determine the helix angle of gear.
4. Determine the pitch diameter of pinion.

Data for Problems 5 to 10:
Center distance = S = variable
Diametral pitch = 7
Number of teeth in gear = 56
Number of teeth in pinion = 28
Shaft angle = 90°

VARIABLE

1. $S = 8.4196$	2. $S = 8.4156$	3. $S = 8.3806$
4. $S = 8.3652$	5. $S = 8.3782$	6. $S = 8.3372$

5. Determine the helix angle of pinion.
6. Determine the pitch diameter of gear.
7. Determine the chordal thickness of pinion.
8. Determine the corrected addendum of pinion.
9. Determine the outside diameter of pinion.
10. Determine the outside diameter of gear.



For Problems 11 to 24, refer to the above figure.

Data for Problems 11 to 18:
Center distance $= J =$ variable
Diametral pitch $= 8$
Number of teeth in gear $= 25$
Number of teeth in pinion $= 15$
Shaft angle $= 50°$
Both gears are left-hand spirals.

VARIABLE

1. $J = 2.7571$	2. $J = 2.7552$	3. $J = 2.7503$
4. $J = 2.7484$	5. $J = 2.7500$	6. $J = 2.7526$

11. Determine the helix angle of the gear.
12. Determine the helix angle of the pinion.
13. Determine the pitch diameter of the gear.
14. Determine the pitch diameter of the pinion.
15. Determine the outside diameter of the gear.
16. Determine the outside diameter of the pinion.
17. Determine the chordal thickness of the gear.
18. Determine the corrected addendum of the gear.

Data for Problems 19 to 24:
Center distance $= K =$ variable
Diametral pitch $= 10$
Number of teeth in gear $= 45$
Number of teeth in pinion $= 15$
Shaft angle $= 60°$
Gear is right-hand spiral, and pinion is left-hand spiral.
Consider the pinion to be the driver, which means that the pinion is to have the greater helix angle.

VARIABLE

1. $K = 6.7922$	2. $K = 6.4265$	3. $K = 6.1149$
4. $K = 5.8458$	5. $K = 5.6112$	6. $K = 5.4049$

19. Determine the helix angle of the gear.
20. Determine the helix angle of the pinion.
21. Determine the pitch diameter of the gear.
22. Determine the pitch diameter of the pinion.
23. Determine the lead of the gear.
24. Determine the lead of the pinion.

DUPLICATING SPIRAL GEARS

The problem of duplicating a set of spiral gears is different from and more difficult than that of designing a set. The following will illustrate the procedure when duplicating a set of spiral gears.

After the gears have been removed from the machine, check the center distance with a pair of micrometers and check the shaft angle with a bevel protractor. Care should be taken accurately to check the shaft angle because the determination of other quantities depends on it.

Next determine the pitch of the gears. Since the pitch used in spiral gearing is normal pitch, it is necessary to measure along a line perpendicular to the teeth. One of the best methods is to cover the tips of the teeth with prussian blue, or some similar substance, and roll the gear along the edge of a ruler on a piece of paper. The impression made will be very similar to that in the following diagram.

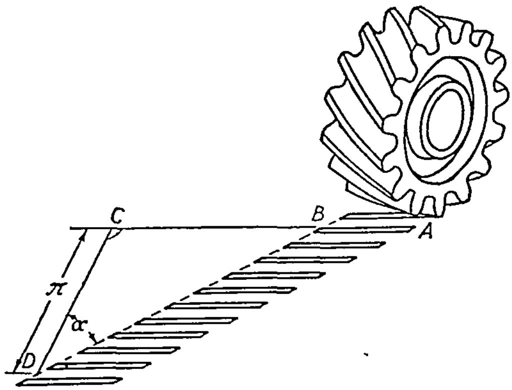


FIG. 160.

Prolong the impression mark AB of one of the teeth. On this prolongation BC erect a perpendicular CD, π in. in length, having one extremity D on the line BD along which the gear has been rolled. Since the diametral pitch of a spiral gear is the number of teeth in π in. along a line perpendicular to the teeth, the diametral pitch may be determined by counting the number of teeth along CD which is the same as the number of teeth within BD.

Since the gear has been rolled on its outside circumference, and since the diametral pitch should be measured on the pitch circumference, the counted number of teeth will be less by a small fraction than the actual diametral pitch. For this reason

the actual diametral pitch will be the next whole number to the number counted. In Fig. 160, the counted number along BD is approximately $9\frac{1}{2}$, which means that the actual diametral pitch is 10.

Note: Be sure to count the teeth between corresponding points.

Count the number of teeth in each gear and measure the center distance. Measure the shaft angle. Measure the helix angle α with a protractor from the sheet upon which the gear was rolled. This angle will correspond to angle CDB in Fig. 160. The angle α can also be determined by obtaining its cosine, which is π divided by the measured distance BD.

Substitute the values of the assumed diametral pitch, the counted number of teeth, the measured center distance, and the shaft angle, in the helix-angle formula,

$$R \sec \alpha_g + \sec \alpha_p = \frac{2CP_n}{n}.$$

If after substitution, the left-hand member of this formula does not approximately equal the right-hand member, the spiral gears are evidently based on circular pitch, not on diametral pitch. In this case determine the circular pitch (which varies by sixteenths) closest to the result of π divided by the assumed diametral pitch.

Usually the two members of the helix-angle formula will be approximately equal after substituting the known values. This means that the assumed diametral pitch is correct. The slight difference between the two members is due to the fact that the helix angle can be measured only approximately. The accurate determination of the helix angle can best be explained by the following illustrative example:

The counted numbers of teeth in the spiral gears are 15 and 12, respectively. The measured center distance is 2.9185 and the measured shaft angle is 124°. Drop a perpendicular π in. long from the prolongation of one of the teeth, and having one of its extremities in a line formed by the ends of the teeth of the gear when rolled. The angle between the line formed by the ends of the teeth and the perpendicular

is equal to the helix angle. With a protractor this angle measures approximately 64°. The number of teeth counted from the tooth of prolongation to where the perpendicular intersects the angle formed by the ends of the teeth is 9 and a fraction. Hence, the diametral pitch is probably 10.

Substituting the foregoing data in the helix-angle formula (starting with the measured helix angle of the gear and not one-half the shaft angle):

$$R \sec \alpha_g + \sec \alpha_p = \frac{2CP_n}{n}.$$

$$1.25 \sec 64° + \sec 60° = \frac{2 \times 2.9185 \times 10}{12}.$$

$$4.8515 = 4.8641.$$

This shows that the left-hand member is not quite equal to the quotient. Therefore, the helix angle must be changed slightly. Letting $\alpha_g = 65°$ (and hence $\alpha_p = 59°$),

$$1.25 \sec 65° + \sec 59° = 4.8993.$$

Thus the quotient (4.8641) is between the results obtained by using $\alpha_g = 64°$ and 65°, respectively.

Compute the number of minutes by interpolation as follows:

$$\begin{array}{rl} 4.8641 & = \text{quotient} \\ \underline{4.8515} & \text{for } \alpha_g = 64° \\ 126 & = \text{difference} \end{array} \qquad \begin{array}{rl} 4.8993 & \text{for } \alpha_g = 65° \\ \underline{4.8515} & \text{for } \alpha_g = 64° \\ 478 & = \text{difference for } 1° \end{array}$$

$$\frac{126}{478} \times 60 = 15.8' \text{ or } 16'$$

Substituting 64° 16′ for α_g and 59° 44′ for α_p:

$$1.25 \sec 64° \, 16' + \sec 59° \, 44' = 4.8630.$$

For $\alpha_g = 64° \, 17'$:

$$1.25 \sec 64° \, 17' + \sec 59° \, 43' = 4.8637.$$

This result is still less than the quotient. Therefore let $\alpha_g = 64° \, 18'$;

$$1.25 \sec 64° \, 18' + \sec 59° \, 42' = 4.8643.$$

Similarly, interpolating for the number of seconds:

$$\begin{array}{ll} 4.8641 = \text{quotient} & 4.8643 \text{ for } \alpha_g = 64^\circ\ 18' \\ \underline{4.8637} \text{ for } \alpha_g = 64^\circ\ 17' & \underline{4.8637} \text{ for } \alpha_g = 64^\circ\ 17' \\ \quad 4 = \text{difference} & \quad 6 = \text{difference for } 1' \end{array}$$

$$\frac{4}{6} \times 60 = 40''.$$

Hence

$$\alpha_g = 64^\circ\ 17'\ 40'' \qquad \text{and} \qquad \alpha_p = 59^\circ\ 42'\ 20''.$$

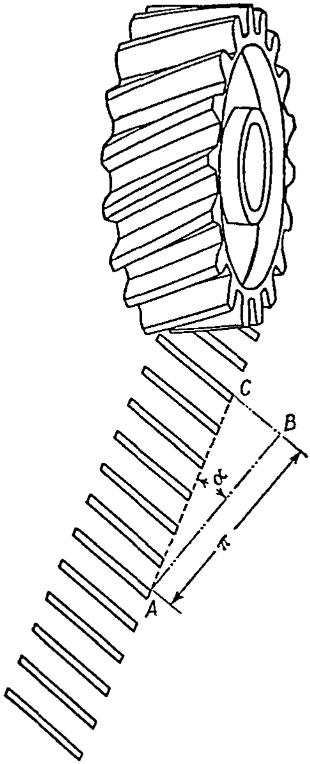
Checking these results by the center-distance formula:

$$C = \frac{N \sec \alpha_g + n \sec \alpha_p}{2P_n} = \frac{15 \sec 64^\circ\ 17'\ 40'' + 12 \sec 59^\circ\ 42'\ 20''}{2 \times 10} = 2.91843,$$

which is approximately equal to the measured center distance of 2.9185.

In case it is not convenient to roll the gears, measure the outside diameter of one of the gears with the micrometer. Determine the helix angles as in the preceding problems by starting with one-half the shaft angle. Then use the formula $D = \dfrac{N \sec \alpha_g}{P_n}$ for determining the pitch diameter and add two addendums to obtain the outside diameter. This should check with the measured outside diameter. There are two pairs of the angles α_g and α_p which will satisfy the helix-angle formula for any given problem. If the pair first obtained does not lead to the correct outside diameter, repeat the work, starting with one-half the shaft angle and varying α_g in the opposite direction.

PROBLEMS



Data for Problems 1 to 5:

Center distance = 4.2919

Number of teeth in gear = 25

Number of teeth in pinion = 18

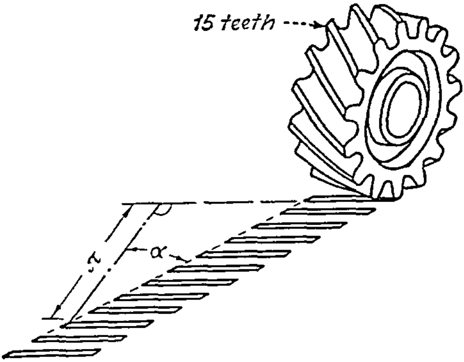
Shaft angle = 63°

Approximate helix angle of pinion = 27°

Both gears are left-hand spirals.

No variable.

1. Determine the helix angle of gear.
2. Determine the helix angle of pinion.
3. Determine the outside diameter of gear.
4. Determine the outside diameter of pinion.
5. Determine the lead of gear.



Data for Problems 6 to 9:

Center distance = 2.9932

Number of teeth in gear = 15

Number of teeth in pinion = 12

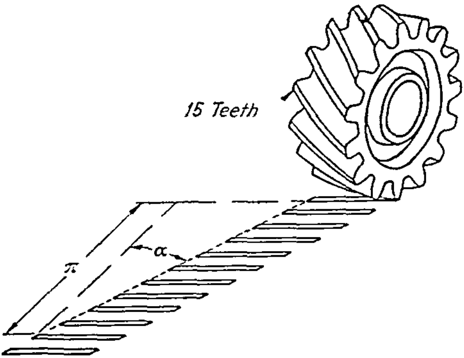
Shaft angle = 120°

Approximate helix angle of gear = 56°30′

Both gears are right-hand spirals.

No variable.

6. Determine the helix angle of gear.
7. Determine the lead of pinion.
8. Determine the outside diameter of gear.
9. Determine the pitch diameter of pinion.



Data for Problems 10 to 13:

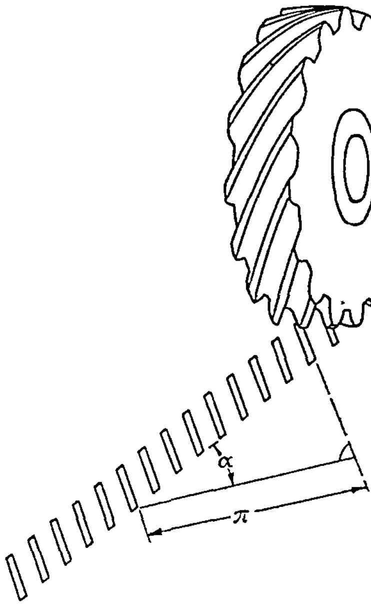
Center distance = 4.6119

Number of teeth in gear = 40

Number of teeth in pinion = 15

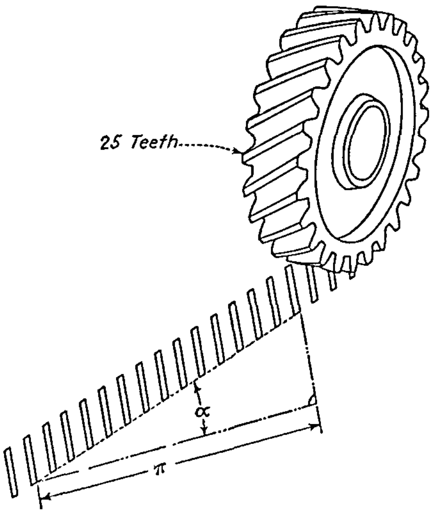
Shaft angle = 105°
Approximate helix angle of gear = 53°
Both gears are right-hand spirals
No variable.

10. Determine the helix angle of gear.
11. Determine the helix angle of pinion.
12. Determine the lead of gear.
13. Determine the outside diameter of pinion.



Data for Problems 14 to 19:
Center distance = 3.9634
Number of teeth in gear = 16
Number of teeth in pinion = 12
Shaft angle = 130°
Approximate helix angle of gear = 60°
Both gears are left-hand spirals
No variable.

14. Determine the helix angle of gear.
15. Determine the helix angle of pinion.
16. Determine the outside diameter of gear.
17. Determine the outside diameter of pinion.
18. Determine the lead of gear.
19. Determine the lead of pinion.



Data for Problems 20 to 25:

Center distance = 1.7717

Number of teeth in gear = 25

Number of teeth in pinion = 20

Shaft angle = 50°

Approximate outside diameter of gear = 2.084

Both gears are left-hand spirals

No variable.

20. Determine the diametral pitch.

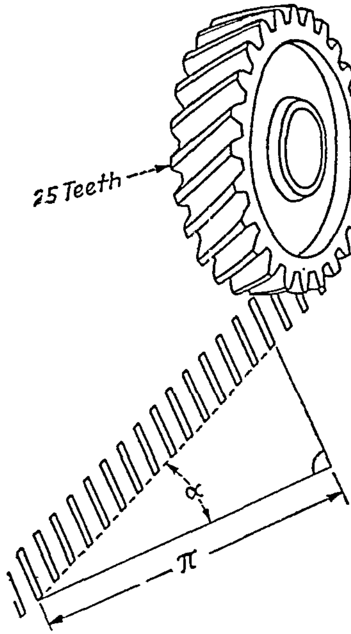
21. Determine the pitch diameter of the gear.

22. Determine the outside diameter of the pinion.

23. Determine the lead of the gear.

24. Determine the helix angle of the gear.

25. Determine the lead of the pinion.



Data for Problems 26 to 31:
- Center distance = 1.5664
- Number of teeth in gear = 25
- Number of teeth in pinion = 18
- Shaft angle = 48°
- Approximate outside diameter of gear = 1.925
- Both gears are left-hand spirals
- No variable.

26. Determine the diametral pitch.
27. Determine the pitch diameter of the gear.
28. Determine the outside diameter of the pinion.
29. Determine the lead of the gear.
30. Determine the helix angle of the gear.
31. Determine the lead of the pinion.

REPLACING SPUR GEARS WITH HELICAL GEARS

There are two main reasons for replacing spur gears with helical gears: (1) to have the gears run more smoothly, consequently producing less noise; (2) to increase the strength of the gear teeth. There are two methods by which any set of spur gears can be replaced by helical gears.

Method 1.—Change the diametral pitch in the spur gear to the next finer normal diametral pitch, which is to be used for the spiral gears. The following is an example of replacing spur gears with helical gears where the gear ratio is to remain unaltered.

Example a: Consider a set of spur gears with 29 and 63 teeth, 8-diametral pitch, to be changed to helical gears, the gear ratio to remain the same.

Solution: The cosine of the helix angle is equal to the original diametral pitch divided by the next finer diametral pitch. In this case it is 8 divided by 9, or .88888. The angle having a cosine of .88888 is 27° 16′. Any set of 8-diametral pitch spur gears can be replaced by 9-diametral pitch helical gears having the same number of teeth and a helix angle of 27° 16′.

Method 2.—Subtract a certain number of teeth from the gear and pinion in accordance with the gear ratio. By using this method the gear ratio in some cases will become slightly changed depending upon the gear ratio itself. When the gear ratio is such that it can be reduced to lower terms, and when in its lowest terms neither of the terms exceed 4, Method 2 can be used without changing the gear ratio. But when the gear ratio is such that it cannot be reduced to a fraction whose terms do not exceed 4, using Method 2 will cause the gear ratio to become slightly changed. When changing from spur gears to helical gears, the ratio of the gears is one of the principal factors and must be considered before making any changes.

Example b: Consider a set of spur gears with 20 and 60 teeth, 7-diametral pitch, to be replaced with helical gears where the gear ratio is to remain the same.

Solution: Since the gear ratio is such that it can be reduced ($\frac{20}{60}$ equals $\frac{1}{3}$; neither term exceeds 4), the change can be made by using Method 2 by subtracting the numerator of the fraction in its lowest terms from the number of teeth in the pinion, and the denominator from the number of teeth in the gear. The number of teeth in the helical gears will then be 19 and 57 teeth, respectively. The cosine of the helix angle of this set of gears is equal to the number of teeth in the helical pinion divided by the number of teeth in the spur pinion which

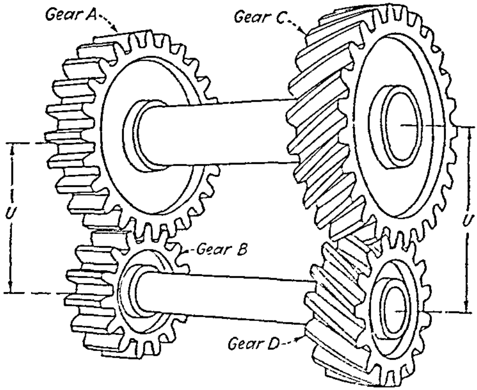
in this case is 19 divided by 20, or .95000. The angle having a cosine of .95000 is 18° 11′.

Example c: Referring to the data of Example *a*, notice that the helix angle is very large but can be used if necessary. In case the gear ratio does not have to be exact, the change can best be made by using Method 2. In this case it will cause the helix angle to be within the required limits of 20°.

Solution: Since the gear ratio is such that it cannot be reduced, choose a fraction approximately equal to the gear ratio. In this case the fraction $\frac{1}{2}$ is approximately equal to $\frac{29}{63}$. Both terms of the approximate fraction are within the required limits of 4. The new numbers of teeth in the helical gears are found by subtracting the numerator of the approximate fraction from the number of teeth in the pinion, and the denominator from the number of teeth in the gear. The numbers of teeth in the helical gears will then be 28 and 61, respectively. The cosine of the helix angle in this case is equal to the sum of the numbers of teeth in the new gears divided by the sum of the numbers of teeth in the old gears, or 89 divided by 92, which is .96739. This is the cosine of 14° 40′.

When the basic reason for changing spur gears to helical gears is to strengthen the gear teeth and the gear ratio can be altered if necessary, Method 2 should be used. When the basic reason for changing from spur gears to helical gears is to have the gears run more smoothly without changing the gear ratio, Method 1 can be used to good advantage. The spiral gears obtained by Method 2 have gear teeth somewhat stronger than the original spur gears.

PROBLEMS



The following eight problems refer to the above figure.

Data for Problems 1 to 4:
Center distance = U = variable.
Gear A has 64 teeth.
Gear B has 23 teeth.
Replace the spur gears with spiral gears, the ratio to be unaltered.

VARIABLE

1. $U = 5.4375$	2. $U = 3.1071$	3. $U = 3.6250$
4. $U = 4.8333$	5. $U = 7.2500$	6. $U = 6.2142$

1. Determine the diametral pitch of spiral gears.
2. Determine the outside diameter of gear D.
3. Determine the lead of gear D.
4. Determine the helix angle of the spiral gears.

Data for Problems 5 to 8:
Center distance = U = variable.
Gear A has 91 teeth.
Gear B has 29 teeth.
Replace the spur gears with spiral gears. Ratio may be altered slightly.

VARIABLE

1. $U = 3.7500$	2. $U = 2.7272$	3. $U = 3.000$
4. $U = 3.3333$	5. $U = 5.000$	6. $U = 4.2857$

5. Determine the diametral pitch of spiral gears.
6. Determine the lead of gear C.
7. Determine the helix angle of spiral gears.
8. Determine the outside diameter of gear D.

MILLING THE CUTTING END OF A SPIRAL FLUTED END MILL

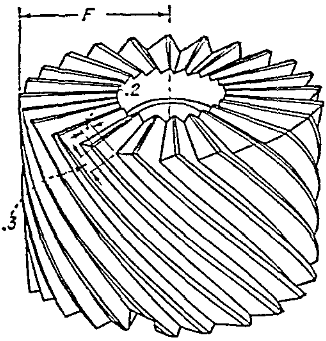


FIG. 161.

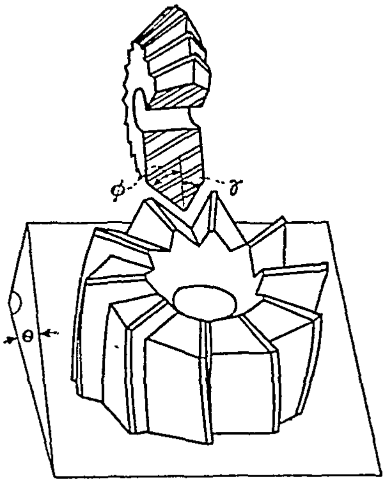


FIG. 162.

VARIABLE

1. $F = 1.500$	2. $F = 1.625$	3. $F = 1.750$
4. $F = 1.875$	5. $F = 2.000$	6. $F = 2.125$

It is desired to cut the end flutes of the spiral fluted end mill shown in Fig. 161. This is accomplished by placing the spiral fluted end mill on a dividing head tilted at an angle θ and using a rotary milling cutter as shown in Fig. 162.

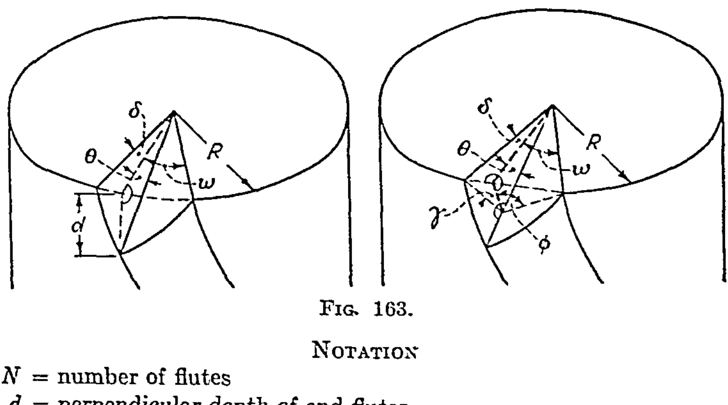


FIG. 163.

NOTATION

N = number of flutes
d = perpendicular depth of end flutes
d' = slant depth of end flutes
D = depth of spiral flute measured on cylinder perpendicular to axis.

F = outside radius
R = root radius
L = lead
α = helix angle (same as helix angle of spiral gear).

Formulas

$$\tan \alpha = \frac{2\pi R}{L}$$

$$d = d' \cos \alpha \text{ (close approximation)}$$

Note: In many spiral fluted end mills the perpendicular depth is given instead of the slant depth.

$$\tan \theta = \frac{d}{R} \qquad \delta = \frac{d}{L}360° \qquad \omega = \frac{360°}{N} - \delta$$

$$\tan \gamma = \tan \delta \operatorname{cse} \theta \qquad \tan \phi = \tan \omega \csc \theta$$

PROBLEMS

The following problems refer to Fig. 161.

Given data:
Variable = outside radius = F
Slant depth = d' = .3
Spiral fluted depth = D = .2
Lead of spiral = L = 9.25

1. Determine the tilting angle θ of the dividing head.
2. Determine the cutting-edge angle γ of the rotary cutter.
3. Determine the cutting-edge angle ϕ of the rotary cutter.

CHAPTER VI

GEAR RATIOS AND LEAD SCREWS

GEAR AND PINION RATIOS

When two shafts are connected by a pair of gears, the numbers of revolutions of the shafts are governed by the numbers of teeth in the gears. The pitch diameters of two gears in mesh have a common point of contact A (indicated by the arrow)

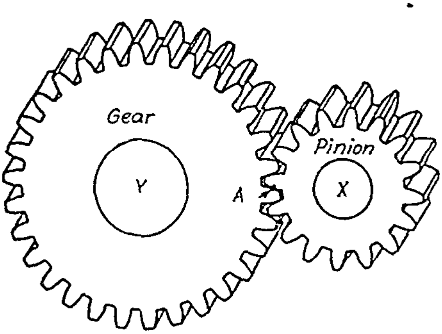


FIG. 164.

In Fig. 164, consider that the gear has 35 teeth and the pinion has 19 teeth. If the gear is revolved 1 revolution, 35 of its teeth will pass the point A. If it makes $\frac{5}{7}$ of a revolution, $\frac{5}{7}$ of 35 (or 25) of its teeth will pass the point A. If it makes $\frac{1}{35}$ of a revolution, $\frac{1}{35}$ of 35 (or 1) of its teeth will pass the point A. Thus when one tooth passes the point A, the gear makes $\frac{1}{35}$ of a revolution. Similarly, if 10 teeth pass the

pinion are intermeshed, it will also cause 35 teeth of the pinion to pass the point A. The number of revolutions made by the pinion is equal to the number of teeth passing point A (35) divided by the number of teeth in the pinion (19) which is $\frac{35}{19}$. It is thus seen that the pinion makes the greater number of revolutions, and the gear the lesser number. Therefore the gear is placed on the shaft which makes the lesser number of revolutions, and the pinion is placed on the shaft that makes the greater number of revolutions.

By the foregoing reasoning it was found that the pinion makes $\frac{35}{19}$ of a revolution while the gear makes 1 revolution. The ratio of the number of revolutions of the pinion to the gear is $\frac{35}{19}$ to 1 (or $\frac{35}{19}$). However, the ratio of the number of teeth in the gear to the number of teeth in the pinion is also 35 to 19 (or $\frac{35}{19}$). Hence the following relation:

$$\frac{\text{Number of revolutions of pinion shaft}}{\text{Number of revolutions of gear shaft}} = \frac{\text{number of teeth in the gear}}{\text{number of teeth in the pinion}}$$

which means that the number of revolutions vary inversely as the number of teeth.

This inverse relation means that the number of revolutions of one shaft may be considered as the number of teeth in the gear on the other shaft and *vice versa.*

Example: Let the shaft X make 41 revolutions and the shaft Y 21 revolutions. In order to have this relation, the gear on shaft Y can have 41 teeth and the pinion on shaft X can have 21 teeth.

Check: The number of teeth of the pinion passing the point A in 41 revolutions is 41×21. Since there are 41 teeth in the gear, the number of revolutions of the gear will be

$$\frac{\text{Number of teeth which pass a point } A}{\text{Number of teeth in gear being revolved}} = \frac{41 \times 21}{41} \text{ or } 21.$$

Increasing or Reducing Gear Teeth (Ratio Unaltered)

The number of teeth in a pair of mating gears can be raised or lowered without changing the ratio of the revolutions of the shafts. This can be done by multiplying or dividing the numbers of teeth in both mating gears by the same

number, in the same manner in which the terms of a fraction are raised or lowered.

Example a: In order to maintain a revolution ratio of 19 to 24, the gear teeth ratio should be 24 to 19. In case a 19 tooth gear is not available, a pair of gears having 2×24 and 2×19, or 48 and 38, teeth may be used. Still another possible set would be 3×24 and 3×19, or 72 and 57 teeth, etc.

Example b: If the revolution ratio of the shafts is 171 to 69, the ratio of the gear teeth will be 69 to 171. A more practical set would be $\frac{69}{3}$ and $\frac{171}{3}$, or 23 and 57, teeth.

SPUR-GEAR AND RACK RATIOS

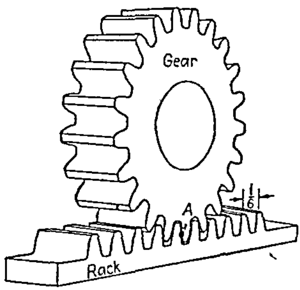


FIG. 165.

Consider that a gear having 17 teeth is in mesh with a rack whose pitch is $\frac{1}{6}$ in. One revolution of the gear will cause 17 teeth of the rack to pass the point of contact A (Fig. 165), thus moving the rack forward seventeen times $\frac{1}{6}$ in. or $2\frac{5}{6}$ in. If the gear makes $\frac{4}{5}$ of a revolution, the rack moves $\frac{4}{5}$ as far or $\frac{4}{5} \times 2\frac{5}{6}$ or $\frac{34}{15}$ or $2\frac{4}{15}$ in.

The reverse of the foregoing is to consider the rack to move forward a certain distance and to compute the corresponding number of revolutions of the gear. Thus, if the rack moves $\frac{5}{6}$ in., the number of teeth of the rack that will pass the point A is $\frac{5}{6}$ divided by $\frac{1}{6}$ or 5. Since five teeth of the gear are thus caused to pass A, the gear will make $\frac{5}{17}$ of a revolution.

Again, if the rack moves forward $1\frac{3}{8}$ in., the number of teeth passing A is $1\frac{3}{8}$ divided by $\frac{1}{6}$ or $8\frac{1}{4}$. Since $8\frac{1}{4}$ teeth of the

gear have passed the point A, the gear will make $\frac{8\frac{1}{4}}{17} = \frac{33}{68}$ of a revolution.

BEVEL-GEAR RATIOS

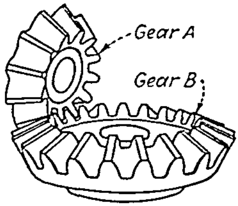


FIG. 166.

In bevel gears, the inverse relation between the numbers of revolutions and the numbers of teeth holds as in the case of spur gears, and thus we may again consider the number of revolutions of either shaft as the number of teeth of the gear on the opposite shaft and *vice versa*. Consider that the bevel gears A and B of Fig. 166 have 16 and 70 teeth, respectively. If gear A makes 1 revolution, gear B will make $\frac{16}{70}$ of a revolution. Conversely, if gear B makes 1 revolution, gear A will make $\frac{70}{16}$ revolutions. If gear B makes 8 revolutions, gear A will make 8 times $\frac{70}{16} = 35$ revolutions.

WORM AND WORM-WHEEL RATIOS

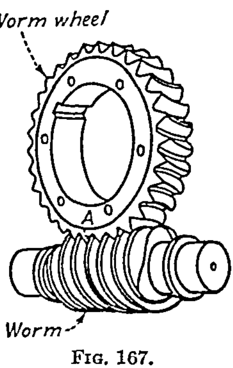


FIG. 167.

In Fig. 167, if the worm is single threaded, one revolution of the worm will cause one tooth of the worm wheel to pass

the point of contact A; if the worm is double threaded, one revolution of the worm will cause two teeth of the worm wheel to pass point A; etc.

Consider that the worm wheel of Fig. 167 has 32 teeth and worm is triple threaded. When the worm makes one revolution, it will cause 3 teeth of the worm wheel to pass point A, or the worm wheel will make $\frac{3}{32}$ of a revolution.

If the worm wheel makes 1 revolution, the worm will make as many revolutions as the number of teeth in the worm wheel contains the number of threads in the worm. If the worm wheel of Fig. 167 makes $\frac{5}{8}$ revolution, the worm will make $\frac{\frac{5}{8} \times 32}{3}$ or $6\frac{2}{3}$ revolutions.

LEAD SCREW AND SLIDE

The lead of a screw is the distance a thread advances along a line parallel to the axis in one complete revolution.

The pitch is the distance from the center of one thread to the center of the next measuring along a line parallel to the axis.

The distance that the slide advances in one revolution of the screw is equal to the lead. On a single-threaded screw, the lead is equal to the pitch; on a double-threaded screw, the lead is equal to twice the pitch; etc.

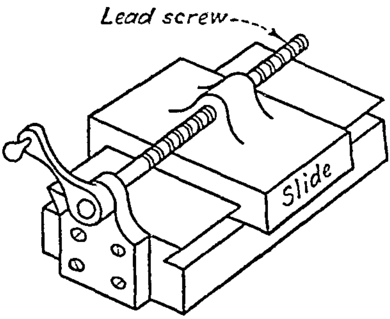


FIG. 168.

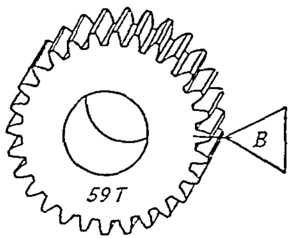
The screw of Fig. 168 is a single-threaded screw and has a pitch of $\frac{1}{7}$ in. Hence the lead is $\frac{1}{7}$ in. and the slide will move forward $\frac{1}{7}$ in. for each revolution of the screw. If the screw makes $3\frac{1}{2}$ revolutions, the slide will move forward $3\frac{1}{2}$ times $\frac{1}{7}$ or $\frac{1}{2}$ in.

If the slide moves $\frac{5}{8}$ in., the screw will make as many revolutions as $\frac{5}{8}$ contains $\frac{1}{7}$, or $4\frac{3}{8}$ revolutions.

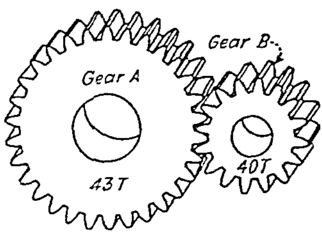
If the screw of Fig. 168 were a triple-threaded screw having a pitch of $\frac{1}{7}$ in., for each revolution of the screw the slide would move forward three times $\frac{1}{7}$ or $\frac{3}{7}$ in. For $5\frac{1}{2}$ revolutions of the screw, the slide will move $3 \times \frac{1}{7} \times 5\frac{1}{2}$, or $2\frac{5}{14}$ inches.

Conversely, if the slide moved $\frac{3}{4}$ in., the screw would make as many revolutions as $\frac{3}{4}$ contains $3 \times \frac{1}{7}$ or $1\frac{3}{4}$ revolutions.

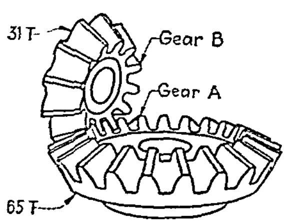
PROBLEMS



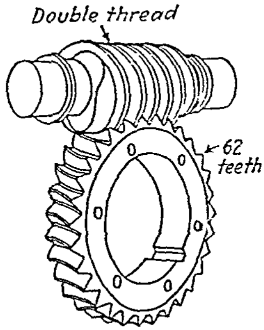
1. When the gear makes G revolutions, how many teeth pass the pointer B?



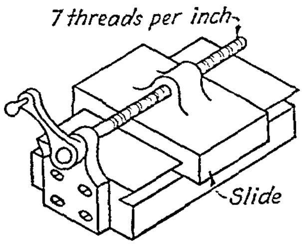
2. When the B gear makes H revolutions, how many revolutions will the gear A make?



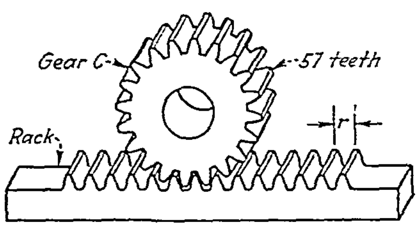
3. When the gear A makes J revolutions, how many revolutions will the gear B make?



4. How many revolutions of the worm will cause the worm wheel to make T revolutions?



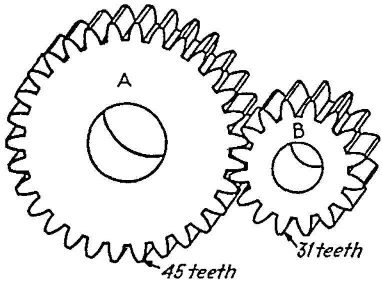
5. How many revolutions must the crank make to cause the slide to move S in.? The fixture above has single thread screw.



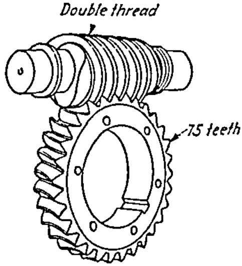
6. When the rack moves 7.532 in., how many revolutions will gear C make?

VARIABLES

Prob.	Sym.	No 1	No 2	No 3	No 4	No 5	No 6
1	G	7 6841	10.305	11 742	13 612	14.735	14.838
2	H	3 405	4.218	5.312	6.687	7.781	8 887
3	J	11.2	12.6	14.5	16.3	17.8	19.5
4	T	1.21	1.78	2.26	2.88	3.42	4.19
5	S	2.115	2.465	2.835	3.223	3.614	4 008
6	r	.375	.5	.4375	.5625	.625	.6875



7. When gear A makes U revolutions, how many revolutions will the gear B make?



8. When the worm makes L revolutions, how many revolutions will the worm wheel make?

VARIABLES

Prob.	Sym.	No. 1	No. 2	No 3	No. 4	No. 5	No. 6
7	U	4 32	5 46	6 18	6 92	7 76	8 56
8	L	21 1	24 2	27 3	31 4	34 5	38 6

IDLER GEARS

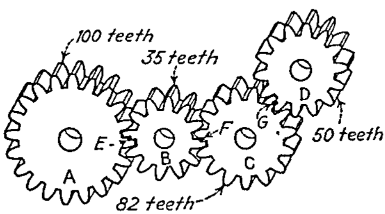


FIG. 169.

Idler gears are gears that mesh with two other gears and are used to transmit power from one shaft to another when

the shafts are too far apart to be connected directly by a pair of gears. Another effect of an idler gear is to change the direction of rotation of the shaft. Idler gears have no effect on the shaft ratio. Thus, in the train of gears *A*, *B*, *C*, and *D* of Fig. 169, when *A* makes 1 revolution, *D* will make as many revolutions as the number of teeth in *A* contains the number of teeth in *D*; *i.e.*, $\frac{100}{50}$ or 2 revolutions. Since *B* and *C* are idler gears, the number of teeth in *B* and *C* are disregarded. The reason for this is as follows:

When *A* makes 1 revolution, 100 teeth will pass the contact point *E*; this will cause 100 teeth to pass the point *F*; this, in turn, will cause 100 teeth to pass *G*, and hence *D* will make $\frac{100}{50}$ or 2 revolutions.

TRAIN OF GEARS

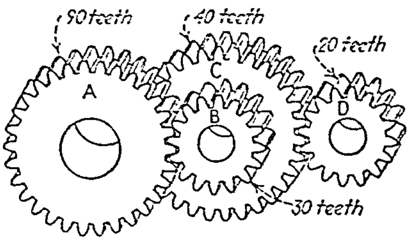


FIG. 170.

In order to determine the number of revolutions that *D* makes when *A* makes one revolution in the train of gears of Fig. 170, obtain the two ratios of the two sets of single gears separately and multiply them together.

Example: When *A* makes 1 revolution, *B* makes $\frac{90}{30}$ or 3 revolutions; when *C* makes 1 revolution, *D* will make $\frac{40}{20}$ or 2 revolutions. Since gears *B* and *C* are keyed to the same shaft, they will make the same number of revolutions. Since *B* and therefore *C* makes $\frac{90}{30}$ revolutions (for 1 revolution of *A*) *D* will make $\frac{90}{30} \times \frac{40}{20}$ or 6 revolutions. Conversely, when *D* makes 1 revolution, *A* will make $\frac{20}{40} \times \frac{30}{90}$ or $\frac{1}{6}$ revolution. When *D* makes $\frac{5}{8}$ of a revolution, *A* will make $\frac{20}{40} \times \frac{30}{90} \times \frac{5}{8}$, or $\frac{5}{48}$ of a revolution.

COMBINATION OF GEARS AND LEAD SCREW

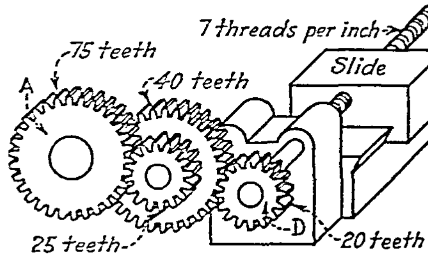


FIG. 171.

Figure 171 shows a train of gears connected with a lead screw and slide. The lead screw is a single threaded screw with seven threads per inch. Consider the problem of determining the distance the slide moves when gear A makes 1 revolution. When A makes 1 revolution, D will make $\frac{75}{25} \times \frac{40}{20}$ or 6 revolutions. When D makes 1 revolution, the slide will move a distance equal to the lead which is $\frac{1}{7}$ in. Since 1 revolution of A causes D to make $\frac{75}{25} \times \frac{40}{20}$ revolutions, and since each revolution of D moves the slide $\frac{1}{7}$ in., the slide will move $\frac{75}{25} \times \frac{40}{20} \times \frac{1}{7}$ or $\frac{6}{7}$ in.

Conversely, if the slide moves $\frac{3}{4}$ in., determine the number of revolutions A makes. Since $\frac{1}{7}$ in. represents 1 revolution of the screw, the gear D, which is attached to the screw, will make as many revolutions as $\frac{3}{4}$ contains $\frac{1}{7}$, or $5\frac{1}{4}$ revolutions. When D makes 1 revolution, A makes $\frac{20}{40} \times \frac{25}{75}$ or $\frac{1}{6}$ revolution. Since D makes $5\frac{1}{4}$ revolutions, A makes $\frac{20}{40} \times \frac{25}{75} \times 5\frac{1}{4}$ or $\frac{7}{8}$ of a revolution.

COMBINATION OF SPUR GEARS AND RACK

In Fig. 172, determine the distance the rack moves when C makes 1 revolution. When A makes 1 revolution, the rack moves a distance equal to the pitch circumference of gear A. The distance from the center of one tooth to the center of the next (on the rack) is equal to the circular pitch which in this case is $\frac{1}{4}$ in. Since there are 20 teeth in gear A, the pitch circumference is equal to $20 \times \frac{1}{4}$ or 5 in., which is the distance

the rack moves when A makes 1 revolution. When C makes 1 revolution, B makes $\frac{25}{35}$ revolution. Gears A and B are keyed to the same shaft and therefore make the same number of revolutions. Therefore, when C makes 1 revolution, the rack will move a distance equal to $\frac{25}{35} \times 20 \times \frac{1}{4}$ or $3\frac{4}{7}$ in.

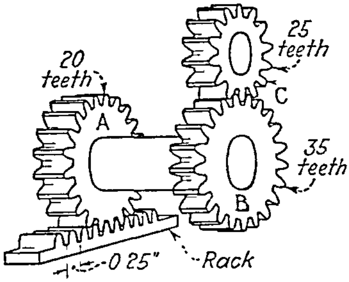


FIG. 172.

Conversely, when the rack moves $\frac{7}{8}$ in., determine the number of revolutions of C. Since the rack moves $\frac{1}{4}$ in. for each tooth, there will be as many teeth passing the point of contact as $\frac{7}{8}$ contains $\frac{1}{4}$, or $3\frac{1}{2}$ teeth. Gear A makes $\frac{3\frac{1}{2}}{20}$ or $\frac{7}{40}$ revolution. Since A makes $\frac{7}{40}$ revolution, C makes $\frac{7}{40} \times \frac{35}{25}$ or $\frac{49}{200}$ revolution.

COMBINATION OF RACK AND SPUR GEAR, AND WORM AND WORM WHEEL

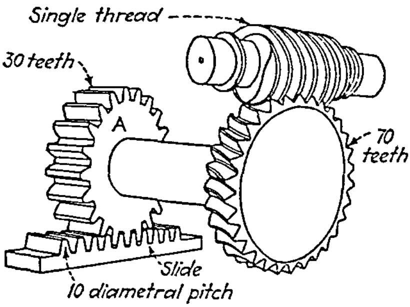
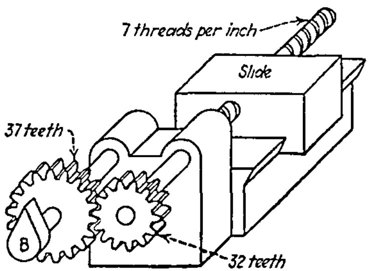


FIG. 173.

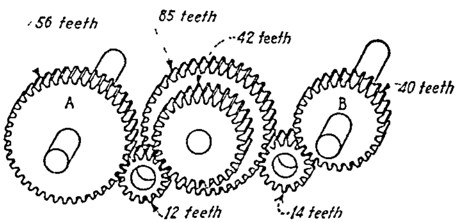
In Fig. 173, determine the distance that the rack moves when the worm makes 1 revolution. The rack is of 10 diame-

tral pitch and the worm is single threaded. When the worm makes 1 revolution, the worm wheel makes $\frac{1}{70}$ of a revolution. Gear A is keyed to the same shaft as the worm wheel and therefore makes the same number of revolutions. Since 1 revolution of A causes the rack to move a distance equal to the pitch circumference of the gear A, $\frac{1}{70}$ of a revolution causes the rack to move $\frac{1}{70}$ as far. The pitch circumference of the gear is equal to $\frac{\pi}{10} \times 30$ or 9.4248. Thus for 1 revolution of the worm the rack will move $\frac{1}{70} \times \frac{\pi}{10} \times 30$ or .13464 in.

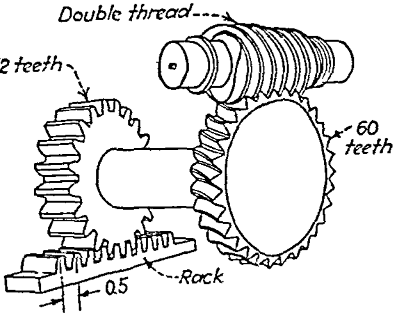
PROBLEMS



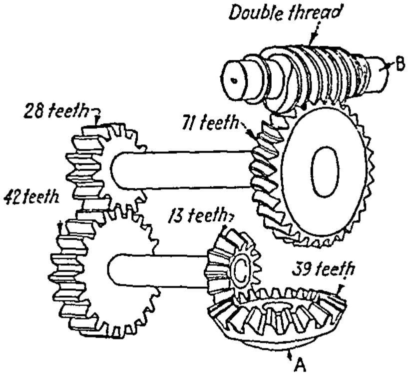
1. When the slide is caused to move G in., how many revolutions will B make?



2. When B makes J revolutions, how many revolutions will A make?
3. How many idler gears are there in this problem?



4. How many revolutions must the worm make in order that the rack will move H in.?

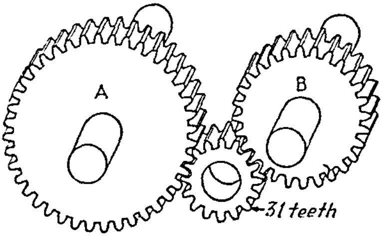


5. When A is caused to make T revolutions, how many revolutions will B make?

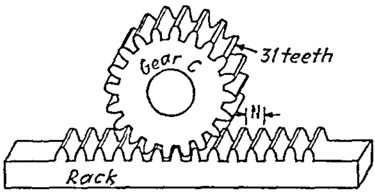
6. When B makes R revolutions, how many revolutions will A make?

VARIABLES

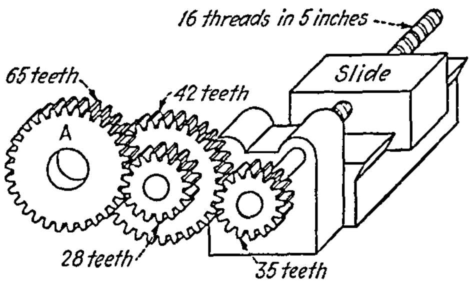
Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	G	*1.328*	*1.514*	*1.815*	*2.315*	*2.468*	*2.735*
2	J	10.3	10.8	11.4	11.9	12.6	13.4
3	J	10.3	10.8	11.4	11.9	12.6	13.4
4	H	2.21	2.46	2.67	2.88	3.05	3.32
5	T	12.5	13.2	13.8	14.5	15.3	15.9
6	R	25.6	27.4	29.3	31.4	33.6	34.5



7. When A makes 5 revolutions, B must make 7. If gear A has S teeth, what number of teeth must B have?



8. When gear C makes 5.73 revolutions, how far will the rack move?



9. When the slide is caused to move P in., how many revolutions will A make?

10. When A makes L revolutions, how far will the slide move?

VARIABLES

Prob.	Sym.	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.	No. 6.
7	S	42	49	56	63	70	35
8	N	.25	.3125	.375	.4375	.5	.5625
9	P	2.68	2 86	2 97	3 12	3 24	3.27
10	L	10.1	12 2	13 3	15 4	17.5	18.6

COMPOUND GEARING

Definitions

The **factors** of a number are the numbers whose product is the number.

Example: 7 and 6 are factors of 42, because 7 times 6 is 42. 3 and 14, and 2 and 21 are also factors of 42.

A **prime number** is a number that has no integral factors, except itself and one.

Example: 2, 3, 5, 7, 11, 13, etc., are prime numbers.

A **prime factor** is a factor that is a prime number.

To find the prime factors of a number, divide the given number by one of the prime numbers in the order given: 2, 3, 5, 7, 11, 13, etc., until you find a number that will divide the given number without a remainder; repeat the same process with the quotients obtained until the final quotient is a prime *number. The continued product of the prime factors equals* the original number.

The prime factors of a number can be rearranged in any order without changing their product.

A number is divisible by 2 when the unit digit is divisible by 2.

A number is divisible by 3 or by 9 when the sum of its digits is divisible by 3 or by 9.

A number is divisible by 5 when the unit digit is 0, or 5.

Use of Factor Table

A factor table should be used when factoring numbers. In order to obtain the factors of a number, first locate the number to be factored. If the number is not a prime number, you will find it in its numerical order in the column headed Number. Immediately to the right of the number in the column headed Factors are the factors of that number. In some books these factors are given with a small number written at the right a little above the number. The small number indicates the power of the number. The power of a number is the number of times a number is taken as a factor. By $3^5 \times 7$ is meant, $3 \times 3 \times 3 \times 3 \times 3 \times 7$ which is equal to 1701.

These prime factors may be arranged in two groups as $3 \times 3 \times 3$ and $3 \times 3 \times 7$. These two groups $3 \times 3 \times 3 = 27$ and $3 \times 3 \times 7 = 63$ are called the composite factors of 1701. The factor table given in the rear of this book gives two factors (neither of which exceeds 120) of all numbers not prime between 20 and 14,400. The two composite factors of a number have been selected as nearly equal to one another as possible. For instance, the composite factors of 225 are given in the table as 15×15 rather than 9×25. In case gears having teeth equal to the given composite factors are not available, rearrange the number into another pair of composite factors for which gears can be obtained.

Compound-gear Arrangement

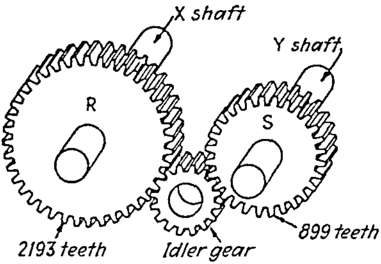


FIG. 174.

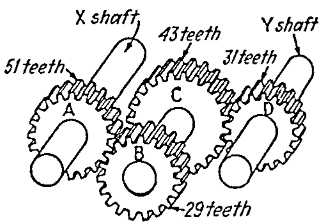


FIG. 175.

Two gears that mesh together causing shafts to revolve are called simple gears. If two or more pairs of simple gears are used, one gear of each pair being placed on a short intermediate shaft called the stud, the train of gears is called compound gears (see Fig. 175). When one intermediate stud is used, the arrangement is called single compound; when two intermediate studs are used, it is called double compound; etc. Compound gears are used on lathes, milling machines, thread-cutting machines, gear-cutting machines, etc., where the number of revolutions of the spindle and lead screw depends upon the distance that the tool must advance.

For ordinary purposes the numbers of teeth in the gears used to transmit the power from one shaft to another are within the limits of 20 and 120. Whenever possible, keep the num-

bers of teeth in the gears between 20 and 100. When the numbers of teeth in the simple gears are such that they cannot be reduced within the required limits, it becomes necessary to arrange the gears in compound order. The driving gear is that gear which transmits the power to the driven gear. In a set of gears, whether simple or compound, the gears are distinguished as being either drivers or driven. Before arranging simple gears in compound order, decide which are drivers and which are driven gears. Either gear of a simple set may be called the driver, the other being called the driven. In arranging simple gears in compound order, the driver must remain the driver and the driven must remain the driven throughout the entire discussion. The factors of the number of teeth of the driver in the simple set become the numbers of teeth of the drivers in the compound arrangement, and likewise for the driven gears.

Rules for Arranging Simple Gears in Compound Order

Rule 1.—*Place the numbers of teeth in the simple gears in the form of a fraction, letting the number of teeth in the driver be the numerator, and the number of teeth in the driven gear be the denominator.*

Rule 2.—*Factor the number of teeth in the driving gear. These factors represent the numbers of teeth in the driving gears of the compound arrangement.*

Rule 3.—*Factor the number of teeth in the driven gear. These factors represent the numbers of teeth in the driven gears of the compound arrangement.*

Rule 4.—*Place one of the driving gears obtained by factoring on the driving shaft and one of the driven gears obtained by factoring on the driven shaft. Either driving gear may be placec on the driving shaft; likewise either driven gear may be placed on the driven shaft.*

Rule 5.—*Place the two remaining gears on the intermediate stud in such a way as to have a driving gear mesh with a driven gear and a driven gear mesh with a driving gear.*

Illustrative Example:

Arrange the simple gears R and S of Fig. 174, in compound order as in Fig. 175. Assume that the X shaft is the driver, and Y the driven shaft. Since X is the driving shaft, R must be the driving gear, and S the driven gear. Assume that R and S have 2193 and 899 teeth, respectively.

Solution: Carrying out the rules in the order given:

Rule 1.—The fraction is $\frac{2193}{899}$.

Rule 2.—The factors of 2193 are 43 and 51.

Rule 3.—The factors of 899 are 29 and 31.

The problem so far may be stated: $\frac{2193}{899} = \frac{43}{29} \times \frac{51}{31}$.

Note: The factors which really represent the numbers of teeth in the gears, for convenience, will be referred to as the gears themselves.

Rule 4.—Since 51 is a driving gear, it may be placed on the shaft X; and since 31 is a driven gear, it may be placed on the shaft Y.

Rule 5.—29 and 43 are the two remaining gears and must be placed on the intermediate stud so that driving gears will mesh with driven gears. Thus 29 must mesh with 51 and 43 with 31 as in Fig. 175.

In general, the four gears used in compound gearing can be arranged in four different ways. In practice, use the most convenient arrangement. In the foregoing problem, a second arrangement is as follows:

A has $43T$, B has $29T$, C has $51T$, and D has $31T$. The student may determine the other two arrangements.

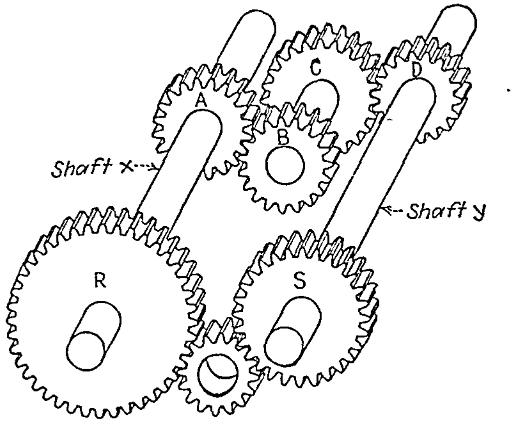
PROBLEMS

Find the prime factors of the following numbers:

1. R.	2. S.	3. T.

Determine from the factor table two factors whose product is equal to the following numbers:

4. J.	5. U.	6. N.
7. M.	8. L.	9. E.
10. P.	11. W.	12. F.



In the following problems determine the numbers of teeth in A, B, C, and D keeping the numbers between the limits of 20 and 100. The speed of A and D is to be the same as R and S, respectively. The numbers below corresponding to R and S are the numbers of teeth in the gears.

13. $R = 1271$, $S = H$. **14.** $R = 3589$, $S = G$.
15. $R = 4189$, $S = K$. **16.** $R = 4087$, $S = V$.
17. $R = 7081$, $S = Q$. **18.** $R = 8051$, $S = Z$.

VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	R	319	377	403	473	437	451
2	S	153	273	351	423	387	399
3	T	513	531	567	663	621	637
4	J	4819	4891	5037	5183	5063	5141
5	U	5917	5963	6059	6497	6241	6319
6	N	3953	4187	4067	4047	4087	4171
7	M	4757	4779	4819	5141	4891	5041
8	L	3139	3233	3283	3431	3337	3403
9	E	1189	1239	1247	1333	1269	1281
10	P	1817	1829	1843	1917	1869	1887
11	W	2419	2449	2451	2499	2457	2491
12	F	2583	2597	2601	2701	2607	2673
13	H	713	729	777	841	819	837
14	G	2343	2403	2449	2537	2479	2499
15	K	2747	2829	2847	2923	2871	2881
16	V	3127	3139	3233	3363	3239	3283
17	Q	6399	6497	6499	6789	6557	6693
18	Z	1357	2449	3127	6319	4757	5429

Raising Gear Teeth Numbers within Proper Limits

In most cases, after the simple gears have been factored and rearranged in compound order, the factors are not within the required limits of 20 and 100. In order to bring these factors within the required limits, one of the factors of the driving gear and one of the factors of the driven gear must be multiplied by some suitable constant. The value of this constant is selected according to the nature of the problem. Sometimes it is necessary to multiply the same factor by two or three different constants in order to obtain suitable numbers of teeth in the gears.

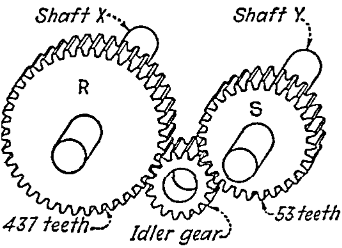


FIG. 176.

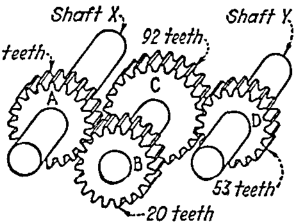


FIG. 177.

Example: In Fig. 176 assume that the gear S has 53 teeth, and the gear R 437 teeth. Also assume the gear S to be the driving gear, and the gear R the driven gear. These gears are to be arranged in compound order as in Fig. 177.

Solution: Applying the first three rules:

Rule 1: $\frac{53}{437}$. Rules 2 and 3: $\frac{53}{437} = \frac{53 \times 1}{19 \times 23}$.

Note: 53 is a prime number, therefore 53 and 1 are the factors of 53.

Before Rule 4 is considered, the factors must be brought within the required limits. Since one of the factors of the driving gear is equal to 1, this factor and one of the factors of the driven gear must be multiplied by a constant whose value is as great as possible. Care should be taken that the constant is not so great that the product of the constant and one of the factors of the driven gear will exceed 100. In selecting the

most suitable constant, notice that 19 can be multiplied by 5 and still be within the limit of 100. Therefore, 5 will be the first suitable constant to use. Then 5 times 1, and 5 times 19 are 5 and 95, respectively. The entire expression so far may be stated thus: $\frac{53 \times 5}{95 \times 23}$. Notice that one of the factors is still below the limit and must be multiplied by another constant. In selecting the second constant, notice that 23 can be multiplied by 4 and still be within the limit of 100. Hence 4 will be the second suitable constant to use. Then 4 times 5 and 4 times 23 are 20 and 92, respectively. Now the entire expression may be stated thus: $\frac{53 \times 20}{95 \times 92}$. These steps placed in a compact form are:

$$\frac{53}{437} = \frac{53 \times 1}{19 \times 23} = \frac{53 \times 5}{95 \times 23} = \frac{53 \times 20}{95 \times 92}.$$

The steps taken, in order to bring the factors within the required limits, are justified by the principle of multiplying the numerator and denominator of a fraction by the same number, which does not change the value of the fraction. Therefore, $\frac{53}{437} = \frac{53 \times 20}{95 \times 92}$. After the factors are within the required limits, proceed according to Rules 4 and 5.

Efficiency of Compound Gears

In a train of gears, the greatest efficiency is produced when the numbers of teeth in the drivers, as well as those in the driven gears, are as nearly equal as possible. In some cases an unsuccessful pair of simple gears can be made more efficient by substituting compound gears. This depends upon the gear ratio. If the numbers of revolutions of the gears are comparatively 8 to 1 or greater, the highest efficiency can be obtained only by arranging the gears in compound order.

PROBLEMS

Referring to Figs. 176 and 177, determine the numbers of teeth in A, B, C, and D for the following values of R and S:

1. $R = 437$, $S = N$.
2. $R = 357$, $S = M$.
3. $R = 899$, $S = L$.
4. $R = 2679$, $S = E$.
5. $R = 35$, $S = P$.
6. $R = 5301$, $S = W$.

Note: When the numbers of revolutions are given in decimal form as in the following two problems, consider the numbers of revolutions to be the numbers of teeth of the gears on the opposite shafts. Carry out Rule 1 and make the numerators and denominators, of the fractions thus obtained, whole numbers by multiplying both numerator and denominator by the proper multiple of 10.

Determine the numbers of teeth of A, B, C, and D according to the numbers of revolutions given for the X and Y shafts (referring to Figs. 176 and 177):

7. X shaft makes K revolutions,
Y shaft makes 11.27 revolutions.
8. X shaft makes 3 revolutions,
Y shaft makes F revolutions.

VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	N	357	323	319	377	299	247
2	M	143	187	209	221	253	273
3	L	731	713	703	697	667	689
4	E	1891	1911	1917	1943	1947	1961
5	P	32	28	25	21	18	15
6	W	4977	4941	4899	4779	4743	4851
7	K	8.05	8.14	7.93	7.95	7.54	7.31
8	F	3.91	4.02	4 13	4.24	4 35	2.66

CHAPTER VII

PLANETARY GEARING

GENERAL TYPES OF PLANETARY GEARING

Planetary gearing, sometimes known as **epicyclic gearing,** consists of a combination of gears meshing together, one of these gears being a stationary gear connected by an intermediate gear or gears to a gear called the planetary gear which is mounted at the end of a rotating arm.

There are three kinds of planetary-gear arrangements commonly used, *viz.*, a single planetary and two types of compound planetary.

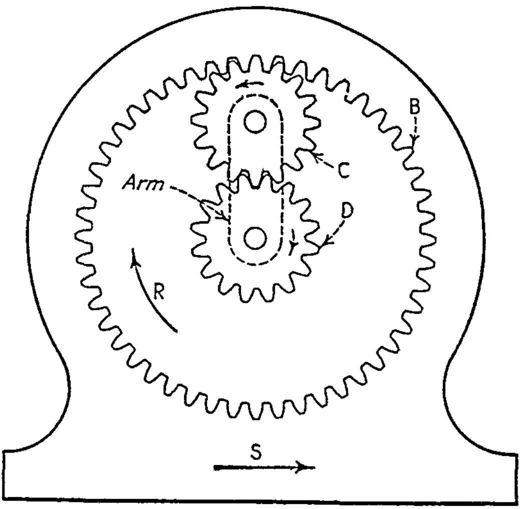


FIG. 178.

In the single planetary arrangement, shown in Fig. 178, the planetary gear revolves within a stationary internal gear and around a central gear, called the sun gear. It follows that the planetary gear must rotate on its own axis.

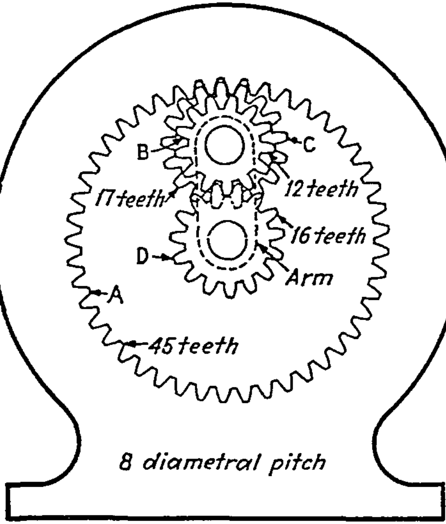


FIG. 179.

In one of the compound planetary arrangements, the sun gear, instead of being connected to the stationary internal gear by means of a single planetary gear, is connected by means of a compound planetary as in Fig. 179.

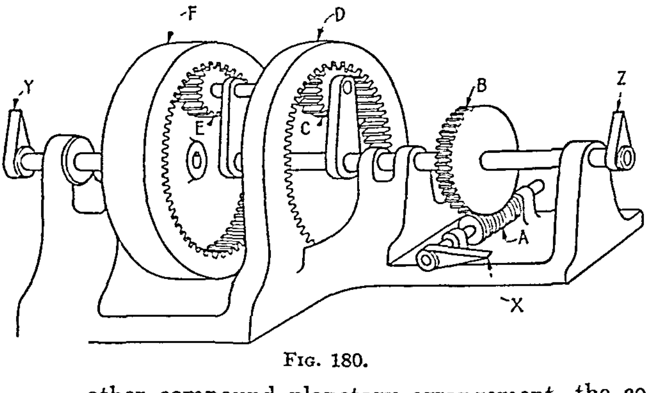


FIG. 180.

other compound planetary arrangement, the compound planetary gears rotate within two separate internal gears, one of which is stationary and the other of which is

caused to rotate by the rotating planetary gear and the rotating arm which is connected to the axis of the rotating planetary gears and the main central axis. The principle of this arrangement is shown in Fig. 180. The theory of this arrangement is given on page 271.

The planetary-gear arrangements offer many possibilities of securing high ratios, which may be used to reduce speeds, by means of few gears having a minimum amount of friction, the gears being placed in a compact form. Planetary gears are usually used in conveyor systems and in some types of lathes and in certain types of special machinery.

THEORY OF PLANETARY GEARING

With reference to Fig. 178, let it be required to determine the number of revolutions that the sun gear makes when the arm makes 1 revolution.

Consider the entire system of gears to be locked and to revolve 1 revolution in the direction of the arrow *R*. Next think of the gears free to rotate but the arm fixed. Now bring the stationary gear back to its original position by revolving it 1 revolution in the opposite direction as indicated by the arrow S. Consider the effects of these two motions on the arm and on the sun gear.

The first motion caused the arm to make 1 revolution in the direction of the arrow *R*. The second motion caused the sun gear to rotate an additional number of revolutions in the same direction (as shown by the small arrows). The additional number of revolutions can be obtained by dividing the number of teeth of the stationary internal gear *B* by the number of teeth in the sun gear *D*, the planetary gear acting as an idler gear. Thus the total number of revolutions of the sun gear, when the arm makes 1 revolution, is $1 + \frac{B}{D}$ where *B* and *D* are the numbers of teeth in the *B* gear and *D* gear, respectively.

The number of teeth in the stationary internal gear is always equal to twice the number of teeth in the planetary gear plus the number of teeth in the sun gear.

In Fig. 178, gears C and D are equal, and hence the number of teeth in B is three times the number of teeth in D or $B = 3D$. Substituting the equivalent of B in the formula $1 + \frac{B}{D}$, the sun gear makes $1 + \frac{3D}{D}$ or 4 revolutions for 1 revolution of the arm. Since the sun gear makes 4 revolutions to 1 revolution of the arm, $4D$ teeth (of the sun gear) correspond to 360° of revolution of the arm, or the advancement of one tooth of the sun gear corresponds to $\frac{360°}{4D}$ or $\frac{90}{D}$ degrees of revolution of the arm. The advancement of one-third of a tooth of the sun gear would correspond to one-third of $\frac{90°}{D}$ or

$$\frac{30}{\text{number of teeth of sun gear}}$$

degrees of revolution of the arm, etc.

PROBLEMS

In Fig. 178, the gears B, C, and D have the numbers of teeth indicated in the chart below.

Prob.	B	C	D
1	36	12	12
2	38	13	12
3	40	14	12
4	42	15	12
5	44	16	12
6	46	17	12

Determine the revolutions of the arm in terms of degrees, minutes, and seconds for a revolution of D equal to one-third of a tooth.

Effect of Planetary Gears on Angles between Spider Arms

In practice a single planetary gear as shown in Fig. 178 is seldom used, but a discussion of it has been necessary to introduce the theory involved in planetary gearing. The actual arrangement consists of three or four (usually three) planetary

gears connected together, mounted at the ends of three or four arms forming a single unit which is called a spider.

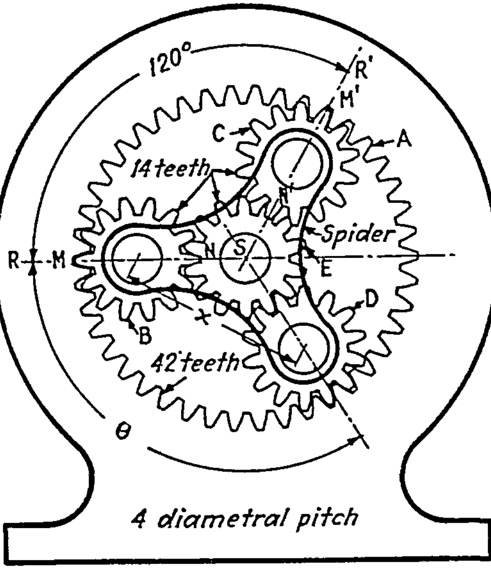


Fig. 181.

A common arrangement is shown in Fig. 181, and to give a better conception of the formation of the spider a top view is shown in Fig. 182.

In order to offset the effect of centrifugal force, it is desirable to have the arms of the spider make equal angles with one another as nearly as possible. Thus in a three-arm spider, the angles between the arms should be 120° or as close to that as is possible. If both the numbers of teeth in the sun gear and the stationary internal gear are evenly divisible by 3, the spider arms will make exactly 120° with one another; but if either or both of the numbers of teeth in the sun and internal gears are not evenly divisible by 3, the angles between the spider arms cannot be 120°. In this case the angles should be made as nearly 120° as possible and must be determined as in the following discussion.

In Fig. 181, there are 42 teeth in the stationary internal gear and 14 teeth in the sun gear and 14 teeth in each of the

planetary gears, all gears being of 4-diametral pitch. The distance between the centers of successive teeth is the same as the distance between the centers of successive spaces.

Along the center line RS, a tooth of a planetary gear fits in a space of the stationary internal gear at M; and since the planetary gear B has an even number of teeth (14), there will be another tooth at N which is directly opposite M on the line RS. The angle RSR' is taken as 120° to begin with so the number of spaces of the internal gear from M to M' is one-

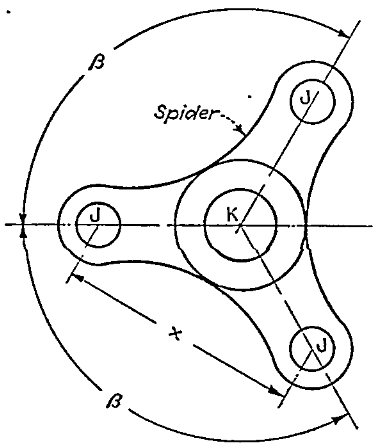


FIG. 182.
JK = center distance of planetary and sun gears.

third of 42 or 14. Thus there is a tooth space at M'. Consider a planetary gear C having a tooth in this space. As before, since there are an even number of teeth in C, there will be a tooth at N'.

The sun gear E must have a space at N in order to mesh with the planetary gear C. Next determine the position of the tooth space in the sun gear E with respect to the center line $R'S$. This is done by dividing the total number of spaces in the sun gear by 3, which in this case is $\frac{14}{3}$ or $4\frac{2}{3}$ spaces. If the number of spaces in the sun gear were evenly divisible by 3, there would have been a tooth space at N'. In this case

the tooth space is two-thirds of a space to the left of the center line $R'S$ which means that the sun gear must be revolved counterclockwise one-third of a tooth space. Note that the teeth in the sun gear, the planetary gear B, and the internal gear are in mesh and that the planetary gear C remains in a fixed position while the sun gear is being rotated.

The effect of rotating the sun gear one-third of a tooth counterclockwise will cause the planetary gear B to rotate clockwise, thus making the line SR recede from the line SR' and hence the angle RSR' is slightly greater than 120°.

The formula for determining the number of degrees of revolution of the spider arm corresponding to a rotation of the sun gear of one-third of a tooth, for the case of the sun gear and planetary gears having the same number of teeth, was shown on page 262 to be $\frac{30°}{\text{number of teeth in sun gear}}$. In this case the value becomes $\frac{30°}{14}$ or 2.1428° which is 2° 8′ 34″. Hence the new angle $R'SR$ between the spider arms is 120° + 2° 8′ 34″ or 122° 8′ 34″.

PROBLEMS

In Fig. 181, the gears A, B, C, D, and E have the numbers of teeth indicated in the chart below:

Prob.	A	B, C, D, and E
7	48	16
8	51	17
9	57	19
10	60	20
11	66	22
12	69	23

Determine the angle β, as nearly as possible to 120° in order that the gears B, C, D, and E when properly placed on the spider arm will fit when inserted in the stationary internal gear.

Determine the distance x.

Discussion of Planetary Gearing for the Numbers of Teeth in Internal and Sun Gears Not Evenly Divisible by the Number of Arms

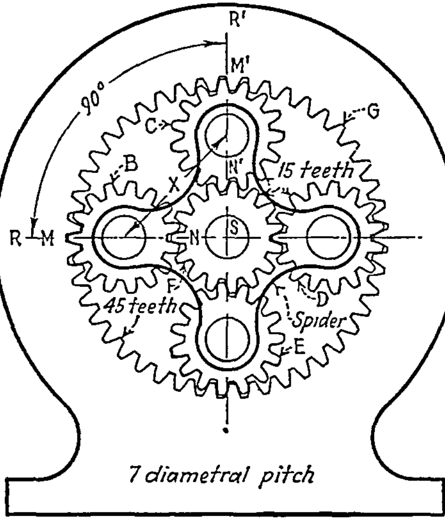


Fig. 183.

In the planetary-gear arrangement shown in Fig. 183, determine whether the angle RSR' can be exactly 90°. The number of spaces in the internal gear divided by the number of arms is $\frac{45}{4}$ or $11\frac{1}{4}$ which, starting with a space at M, means that the eleventh space is one-fourth of a space to the left of the line SR'. It follows from this that the tooth of the C gear is one-fourth of a tooth to the left of the line SR'. Hence, there being an odd number of teeth in gear C, the space opposite this tooth is one-fourth of a space to the right of the line SR'.

The number of teeth in the sun gear divided by the number of arms is $\frac{15}{4}$ or $3\frac{3}{4}$, which, starting with a tooth at N, means that the fourth tooth of the sun gear is one-fourth of a tooth to the right of SR'. Since there is a tooth of the sun gear at one-fourth of a tooth division to the right of SR', and since there is a space of the gear C one-fourth of a space division

to the right of SR', the two gears will mesh perfectly for the angle RSR' equal to 90°. It is always true for a spider of four arms and equal numbers of teeth in planetary and sun gears that the spider arms will be exactly 90° apart.

Next consider the case where the numbers of teeth of the internal and sun gears are not evenly divisible by 4 and the numbers of teeth in the planetary and sun gear are not equal. Assume that the sun gear of Fig. 183 has 13 teeth, the planetary gears 18 teeth each, and the internal gear 49 teeth. Determine the angle RSR' as nearly as possible to 90° in order that the gears will mesh.

Carrying out briefly the same procedure as before: $\frac{49}{4}$ is $12\frac{1}{4}$, which, if it is borne in mind that the planetary gear in this case has an even number of teeth, means that there is a tooth at N' which is one-fourth of a division to the right of the line SR'. Referring to the sun gear, $\frac{13}{4}$ is $3\frac{1}{4}$ which means that there is a space three-fourths of a division to the right of the line SR'. In order that the sun gear and the gear C may mesh, the sun gear must be rotated counterclockwise one-half of a tooth division. This causes the gear B to rotate clockwise, making the new angle $R'SR$ slightly greater than 90°. The student may determine this angle exactly.

PROBLEMS

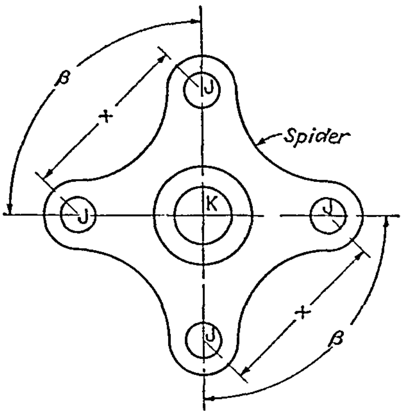


Fig. 184.

Figure 184 shows the spider for the four-arm arrangement of Fig. 183.

JK = center distance of planetary and sun gears.

Consider that the numbers of teeth in the gears A, B, C, D, E, and F are as given in the following chart:

Prob.	A	B, C, D, and E	F
13	41	13	15
14	43	13	17
15	45	13	19
16	47	13	21
17	49	13	23
18	51	13	25

Determine the angle β as nearly as possible to 90°, and the distance x of Fig. 184 so that the gears B, C, D, E, and F of Fig. 183 when properly placed on the spider arm, will fit when inserted in the stationary internal gear.

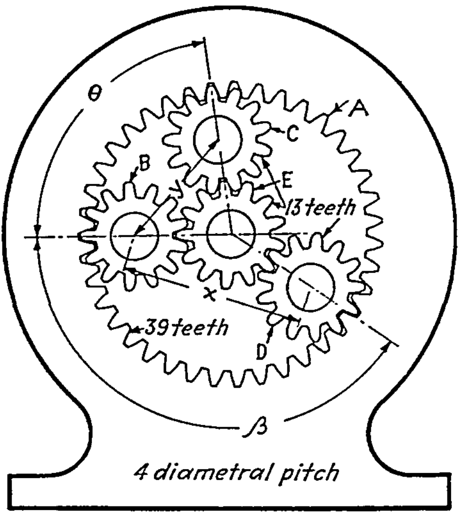


FIG. 185.

In the three-arm planetary-gear arrangements of Fig. 185, having a spider similar to that of Fig. 182, the numbers of teeth in the gears A, B, C, D, and E are as given in the following chart:

Prob.	*A*	*B*, *C*, and *D*	*E*
19	41	12	17
20	43	12	19
21	44	12	20
22	46	12	22
23	47	12	23
24	49	12	25

Determine the angle β as nearly as possible to 120° and the distance x of Fig. 182 so that the gears B, C, D, and E of Fig. 185, when properly placed on the spider arm, will fit when inserted in the stationary internal gear.

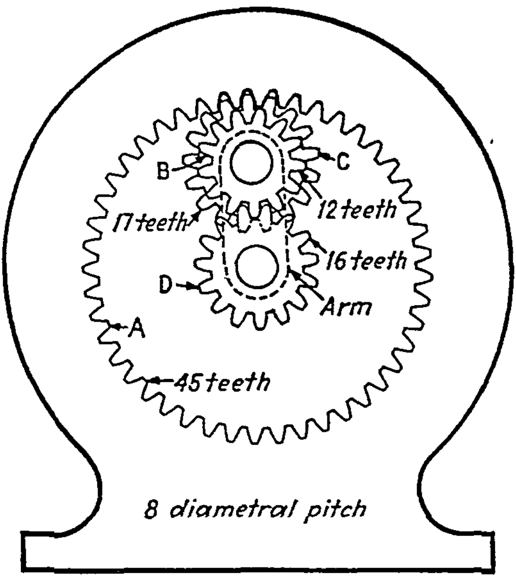


Fig. 186.

For the compound planetary gear arrangement of Fig. 186, determine the revolution of the arm when the sun gear makes a revolution of one tooth. As in the previous discussion for simple planetary gears given on page 261, consider the locked system of gears revolved 1 revolution clockwise, which means that the sun gear makes one revolution clockwise. With the

arm fixed, rotate the internal gear counterclockwise. This causes the sun gear to rotate clockwise $\frac{A}{C} \times \frac{B}{D}$ revolutions, making a total of $1 + \frac{A}{C} \times \frac{B}{D}$ revolutions of the sun gear for 1 revolution of the arm. Thus $\left(1 + \frac{A}{C} \times \frac{B}{D}\right) \times D$ teeth of the sun gear correspond to 360° of the arm, or a revolution of one tooth of the sun gear corresponds to $\dfrac{360}{D + \frac{A}{C} \times B}$ degrees of revolution of the arm.

PROBLEMS

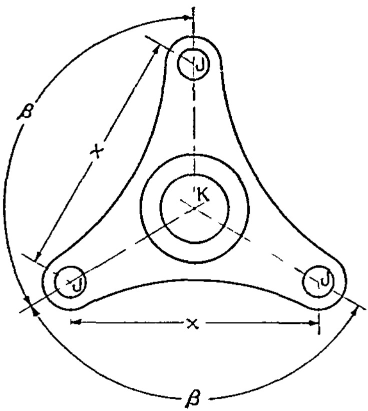


FIG. 187.

Instead of the single-arm compound planetary-gear arrangement shown in Fig. 186, which was used for analytical purposes, in practice there are usually three arms, the spider for which is shown in Fig 187.

The compound gears B and C are keyed together in such a manner as to have the center of a tooth on the C gear in line with the center of a space of the B gear.

The center distance JK of Fig. 187 is the center distance for the gears B and D.

Consider that the numbers of teeth in the gears A, B, C, and D are as given in the following chart:

Prob.	A	B	C	D
25	48	14	18	16
26	51	15	19	17
27	57	17	21	19
28	60	18	22	20
29	66	20	24	22
30	69	21	25	23

Determine the angle β, as nearly as possible to 120°, and the distance x of Fig. 187 so that the gears B, C, and D of each arm of the spider for Fig. 186 when properly placed on the spider arm will fit when inserted in the stationary internal gear.

Compound Planetary Gearing with Arm Acting as Driver, Sun Gear Omitted

The compound planetary-gear arrangement shown in Fig. 180 and described on page 260 may be used to obtain very low velocity ratios.

The following discussion will show the number of revolutions Y will make for 1 revolution of Z: Consider the entire system of gears locked and rotated 1 revolution clockwise. This means that the arm and hence Z has made 1 revolution clockwise. Next think of the gears C, D, E, and F free to rotate but the arm fixed. Now bring the stationary gear back to its original position by revolving it 1 revolution counterclockwise, which causes Y to make $\frac{D}{C} \times \frac{E}{F}$ revolutions in the counterclockwise direction. The first motion (gear locked) caused Y to make 1 revolution clockwise, and hence the total revolutions of Y is $1 - \frac{D}{C} \times \frac{E}{F}$ revolutions which will be either clockwise or counterclockwise according as the product of the ratios $\left(\frac{D}{C} \times \frac{E}{F}\right)$ is less than 1 or greater than 1, respectively. Further to reduce the velocity ratio, a worm and worm wheel may be used as shown in Fig. 180. Then for 1 revolution of X,

$$Y \text{ makes } \frac{A}{B}\left(1 - \frac{D}{C} \times \frac{E}{F}\right) \text{ revolutions.}$$

PROBLEMS

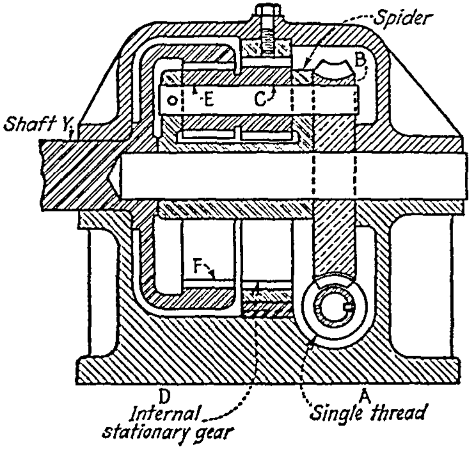


FIG. 188.

In the compound planetary arrangement of Fig. 188, which is similar to that of Fig. 180, consider that the numbers of teeth in the gears and worm wheel are as given in the chart below:

Prob.	*A*	*B*	*C*	*D*	*E*	*F*
31	1	30	23	93	24	97
32	1	40	20	99	19	94
33	1	50	20	81	21	85
34	1	30	18	73	19	77
35	1	40	20	89	18	80
36	1	50	23	97	19	80

Determine the number (expressed decimally) and the direction of the revolutions of the *Y* shaft for 1 revolution of the worm.

CHAPTER VIII

PLAIN AND DIFFERENTIAL INDEXING

PLAIN INDEXING

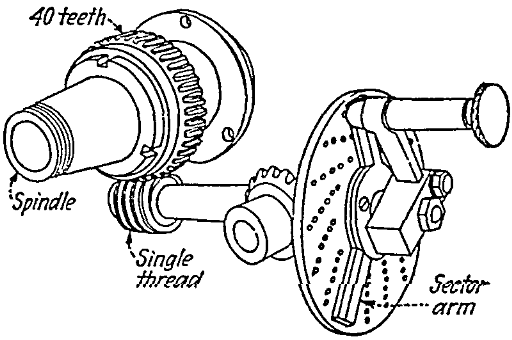


FIG. 189.

The principal working parts of a dividing head are the worm, worm wheel, spindle, crank, sector arm, and index plate, as shown in Fig. 189. The worm and crank are attached to the same shaft, therefore making the same number of revolutions. The worm wheel is attached to the spindle. The index plate is placed on the same shaft with the worm and can be either stationary or movable. For plain indexing the index plate is held stationary with a pin located at the back of the index plate. For differential indexing the index plate is allowed to rotate, either in the same direction as the crank or in the opposite direction, depending upon the nature of the problem. When the number of divisions are such that they cannot be made with plain indexing, they are generally determined by differential indexing. The sector arm is used to indicate the fraction of a turn of the crank required by the given problem and can be adjusted to give any fractional part of a turn.

There are three index plates furnished with a Brown and Sharpe dividing head. The numbers of holes in the different

circles contained by these three index plates are 15-16-17-18-19-20-21-23-27-29-31-33-37-39-41-43-47-49.

For plain indexing, assume that the worm and worm wheel are in the ratio of 40 to 1. When the spindle makes 1 revolution, the crank makes 40 revolutions. Suppose that it is required to cut 12 teeth on a gear. This gear is mounted on a chuck on the spindle of the dividing head which means that the spindle must make $\frac{1}{12}$ revolution when going from one tooth to the next. Hence the crank makes one-twelfth of 40, or $3\frac{1}{3}$ revolutions (meaning that the crank makes three complete turns and one-third of a turn). Mount, on the same shaft with the crank, an index plate in which the number of holes in a circle is evenly divisible by the denominator of the fractional part of a revolution that the crank must make. In this case, any of the possible circles given above, which are evenly divisible by 3, may be used, *i.e.*, 15, 18, 21, 27, 33, and 39. If a circle having 33 holes is used to obtain one-third of a turn, the two-part sector arm must be expanded to one-third of 33 or 11 circular divisions where a circular division is the fractional part of a turn of the crank between the centers of two successive holes. In this case a circular division represents one thirty-third of a turn.

Rules for Plain Indexing

Rule 1.—*Form a fraction having for its numerator the number of turns the crank makes when the spindle makes one revolution, and for its denominator the required number of divisions to be indexed.*

Rule 2.—*Reduce this fraction to a mixed number, or to its lowest terms. When the fraction reduces to a mixed number, the whole number indicates the complete turns that the crank must make, and the fraction (reduced) the fractional part of a turn.*

Rule 3.—*Choose an index plate having a number of holes evenly divisible by the denominator of the fraction obtained by Rule 2. To obtain the number of circular divisions that the sector arm must expand, multiply the number just obtained by the fraction of Rule 2.*

Angular Indexing

One revolution of the crank revolves the spindle $\frac{1}{40}$ revolution, or 9°. When the crank is turned 1 hole in a 9-hole circle, the spindle revolves 1°; when the crank is turned 1 hole in an 18-hole circle, the spindle revolves $\frac{1}{2}$°; when the crank is turned 1 hole in a 27-hole circle, the spindle revolves $\frac{1}{3}$°, etc.

PROBLEMS

The following problems refer to Fig. 189.

1. How many revolutions must the crank make in order that the spindle will make *E* revolutions?

2. When the spindle makes *F* revolutions, the crank makes how many?

3. How many revolutions must the crank make in order that the spindle will make *G* revolutions?

4. When the spindle makes *H* revolutions, the crank makes how many?

5. How many holes must the index circle have in order to index *J* divisions?

6. How many holes must the index circle have in order to index *K* divisions?

7. How many holes must the index circle have in order to index *L* divisions?

8. What number of circular divisions must the sector arm expand in order to index *M* divisions?

9. What number of circular divisions must the sector arm expand in order to index *N* divisions?

10. What number of circular divisions must the sector arm expand in order to index *P* divisions?

Variables

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	*E*	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{7}{8}$
2	*F*	3.5	4.5	5.5	6.5	7.5	8.5
3	*G*	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$\frac{6}{13}$	$\frac{7}{13}$	$\frac{8}{13}$
4	*H*	$\frac{1}{37}$	$\frac{1}{39}$	$\frac{1}{41}$	$\frac{1}{43}$	$\frac{1}{47}$	$\frac{1}{49}$
5	*J*	11	12	13	14	15	16
6	*K*	21	22	23	24	25	26
7	*L*	32	33	34	35	36	37
8	*M*	41	42	43	44	45	46
9	*N*	82	84	85	86	88	90
10	*P*	50	52	54	55	56	58

DIFFERENTIAL INDEXING

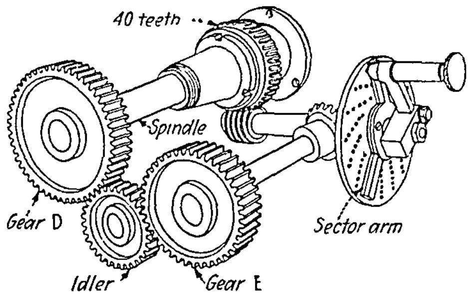


FIG. 190.

Differential indexing is used to index prime and fractional numbers that cannot be obtained by plain indexing.

The procedure for differential indexing can be best illustrated by the following two problems.

Example a: Assume that 73 divisions are to be indexed.

Solution: Applying Rule 1 of plain indexing, the fraction is $\frac{40}{73}$, the denominator of which cannot be reduced to any of the given numbers of holes in the index plates. Therefore, select some number close to 73 which will give a fraction that can be reduced within the range of plain indexing, say 70. If 70 is used, the crank will rotate $\frac{40}{70}$ or $\frac{4}{7}$ revolution each time it is operated. Since 73 divisions must be indexed, the crank is operated seventy-three times while the spindle is supposed to make 1 revolution. The number of revolutions that the crank makes when it is operated seventy-three times is $73 \times \frac{4}{7}$ or $41\frac{5}{7}$ revolutions, but the crank should make 40 revolutions while the spindle makes 1. Therefore, the crank has made $1\frac{5}{7}$ or $\frac{12}{7}$ revolutions too many. This means that the index plate must rotate $\frac{12}{7}$ revolutions in the direction opposite to that of the crank while the spindle makes 1 revolution. This is to be accomplished by means of gears *D* and *E* of Fig. 190, *D* being the gear on the spindle and *E* the gear on the shaft which actuates the gear attached to the index plate. The numbers of revolutions $\frac{12}{7}$ to 1 or 12 to 7 correspond to the numbers of teeth in the gears on the opposite shafts. There-

fore, the numbers of teeth in the gears on the spindle and index plate are in the ratio of 12 to 7, respectively.

Example b: Assume that 461 divisions are to be indexed.

Solution: Select some number very nearly equal to the number of divisions required, which is within the range of plain indexing, *e.g.*, 470. In order to cut 470 divisions, the crank must make $\frac{40}{470}$ or $\frac{4}{47}$ of a turn for each division. Since each time the crank is operated, it rotates $\frac{4}{47}$ of a turn, then when the crank is operated 461 times, it will make $461 \times \frac{4}{47}$, or $39\frac{11}{47}$ revolutions, which is $\frac{36}{47}$ revolution less than it should make. Now the index plate must rotate $\frac{36}{47}$ revolution in the same direction as the crank, while the spindle makes 1 revolution. For the same reason given above, the gears on the spindle and index plate are in the ratio of 36 to 47, respectively, or the gear on the spindle has 36 teeth and the gear on the index plate has 47 teeth.

In each of the foregoing problems, it may be noticed that the gear ratio is equal to the product of the fractional part of a revolution that the crank makes for each division, and the difference between the number of divisions required and the number of divisions selected. This relation can be proved algebraically.

Rules for Differential Indexing

Rule 1.—*Select some number nearly equal to the number of divisions required, which, when divided by one of the factors of* 40 (2, 4, 5, 8, 10, 20, 40), *will give a number that can be indexed by plain indexing.*

Rule 2.—*Form a fraction with* 40 *as the numerator and the selected number as the denominator and reduce it to its lowest terms.*

Rule 3.—*Choose an index plate having a number of holes evenly divisible by the denominator of the fraction obtained by Rule* 2. *Multiply this number by the fraction of Rule* 2 *to give the number of circular divisions that the sector arm must expand.*

Rule 4.—*Multiply the fraction of Rule* 2 *by the difference between the number selected and the number of divisions required. This product is equal to the gear ratio, which, if necessary, may*

be raised to higher terms in order to get numbers which are within the usual range for gear teeth. The numerator and denominator of this fraction are equal to the number of teeth in the gear on the spindle and the number of the teeth in the gear on the index plate, respectively.

PROBLEMS

The following problems refer to Fig. 190.

Notation

A = number of divisions in index circle that the crank must move
B = number of holes in circle to be used
D = number of teeth in gear on spindle
E = number of teeth in gear on worm

The numbers of teeth in gears must be within the limits of 20 and 100.

1. Number of divisions to be indexed equals R. Determine the values of A, B, D, and E.
2. Number of divisions to be indexed equals S. Determine the values of A, B, D, and E.
3. Number of divisions to be indexed equals T. Determine the values of A, B, D, and E.
4. Number of divisions to be indexed equals U. Determine the values of A, B, D, and E.

Variables

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	R	101	107	113	117	121	131
2	S	357	361	373	383	323	347
3	T	563	571	585	592	551	523
4	U	1841	1762	1993	1684	1805	1606

CHAPTER IX

COMBINING FRACTIONS AND CONTINUED FRACTIONS

COMBINING FRACTIONS

A knowledge of continued fractions is valuable and is used principally in the arranging of gears in compound gearing. These gears are placed in hobbing machines, lathes, milling machines, etc., for the purpose of causing the tool to travel a specified distance when the spindle makes one revolution. In the formation of compound gearing, it is understood that both the numerator and the denominator of the fraction involved must be composite numbers. In most cases the numerator and denominator of the fraction involved will not factor; therefore it is necessary to obtain other fractions whose values are a very close approximation of the fraction involved and whose numerator and denominator will factor. The process of deriving other fractions within a specified limit will be explained below.

The fraction derived by adding separately the numerators and denominators of two given fractions is called an intermediate fraction. The fraction derived by subtracting separately the numerator and denominator from the numerator and denominator of another fraction is called an exterior fraction. The two given fractions are called initial fractions. The process by which the intermediate fraction is obtained is called combining fractions by addition. The process by which the exterior fraction is obtained is called combining fractions by subtraction. In order to obtain all the intermediate fractions within a specified limit, combine intermediate fractions already obtained with one another and with the initial fractions until as many as possible intermediate fractions within the imposed limit have been obtained. Care should be taken that the intermediate fractions are always in their lowest terms before forming other intermediate fractions.

In order to obtain all the exterior fractions within a specified limit, first combine the two initial fractions by subtraction, raising one of the initial fractions to higher terms if necessary. Then combine the exterior and initial fraction by addition, in order to obtain (if there are any) the intermediate fractions whose denominators are within a specified limit. Do this continually until as many as possible exterior fractions within the imposed limit have been obtained.

The value of the intermediate fraction is always somewhere between the values of the two given fractions.

Example: For the two initial fractions, $\frac{1}{4}$ and $\frac{3}{8}$, determine the first intermediate fraction.

Solution: The numerator and denominator of the intermediate fraction are composed of the sum of the numerators and denominators of the two given fractions, which in this case is $\frac{4}{12}$, or $\frac{1}{3}$. It may be proved that the value of $\frac{1}{3}$ is somewhere between the values of $\frac{1}{4}$ and $\frac{3}{8}$ by expressing each of the three fractions in terms of twenty-fourths. Thus: $\frac{1}{4} = \frac{6}{24}$, $\frac{1}{3} = \frac{8}{24}$, $\frac{3}{8} = \frac{9}{24}$. The value of the numerator of $\frac{1}{3}$ in terms of twenty-fourths is seen to be greater than the numerator of $\frac{1}{4}$ in terms of twenty-fourths and less than the numerator of $\frac{3}{8}$ in terms of twenty-fourths, and hence the value of $\frac{1}{3}$ is between $\frac{1}{4}$ and $\frac{3}{8}$.

The value of the exterior fraction is either greater than the initial fraction having the greater value or less than the initial fraction having the lesser value. This depends upon the procedure taken. When the numerator and denominator of the initial fraction having the lesser value are subtracted from the numerator and denominator of the initial fraction having the greater value, the resulting exterior fraction is greater in value than the initial fraction having the greater value. When the numerator and denominator of the initial fraction having the greater value are subtracted from the numerator and denominator of the initial fraction having the lesser value, the resulting exterior fraction is less in value than the initial fraction having the lesser value.

Example a: For the two initial fractions, $\frac{1}{4}$ and $\frac{3}{8}$, determine two exterior fractions, one greater in value than $\frac{3}{8}$ and the other less than $\frac{1}{4}$.

Solution: The numerator and denominator of the exterior fraction are obtained by subtracting the numerators and denominators of the two given fractions, which in this case is $\frac{2}{4}$, or $\frac{1}{2}$. In order to determine the exterior fraction less in value than $\frac{1}{4}$, it becomes necessary to raise $\frac{1}{4}$ to higher terms, *e.g.*, $\frac{4}{16}$, then combine by subtraction in the opposite order. The resulting exterior fraction in this case will be $\frac{1}{8}$. It may be proved that $\frac{1}{2}$ is greater than $\frac{3}{8}$ and $\frac{1}{8}$ is less than $\frac{1}{4}$ by expressing each of the four fractions in terms of eighths.. Thus: $\frac{1}{2} = \frac{4}{8}$, $\frac{3}{8} = \frac{3}{8}$, $\frac{1}{4} = \frac{2}{8}$, $\frac{1}{8} = \frac{1}{8}$. The value of the numerator of $\frac{1}{2}$ in terms of eighths is seen to be greater than the numerator of $\frac{3}{8}$ and the value of the numerator of $\frac{1}{8}$ is seen to be less than the value of the numerator of $\frac{1}{4}$ in terms of eighths, and hence the exterior fractions are greater and less, respectively, than the initial fractions.

Example b: Determine all the intermediate fractions of $\frac{1}{4}$ and $\frac{5}{16}$. The denominators of the intermediate fractions are not to exceed 16.

Solution:

A	*B*	*C*	*D*	*E*	*F*	*G*
$\frac{1}{4}$	$\frac{4}{15}$	$\frac{3}{11}$	$\frac{2}{7}$	$\frac{3}{10}$	$\frac{4}{13}$	$\frac{5}{16}$

E is formed by combining *A* and *G* and reducing. *F* is formed by combining *E* and *G* and reducing. *D* is formed by combining *A* and *E* and reducing. *C* is formed by combining *A* and *D*. *B* is formed by combining *A* and *C*.

Example c: Determine all the exterior fractions of $\frac{1}{4}$ and $\frac{3}{8}$. The denominators of the exterior fractions are not to exceed 8.

Solution:

A	*B*	*C*	*D*	*E*	F	G	*H*	*U*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*
$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{1}{1}$

I is formed by combining by subtraction *G* and *F*. *H* is formed by combining by addition *G* and *I*. *U* is formed by combining by addition *H* and *I*. *J* is formed by first raising *I* to $\frac{7}{14}$, then combining by subtraction *I* and *U*. *K* is formed by combining by subtraction *J* and *I*. *L* is formed by first

raising K to $\frac{9}{15}$ and then combining by subtraction K and J. M is formed by combining by subtraction L and K. N is formed by first raising M to $\frac{10}{13}$ and then combining by subtraction M and L. O is formed by combining by subtraction N and M. P is formed by first raising O to $\frac{9}{12}$ and then combining by subtraction O and N. T is formed by combining by subtraction O and P. Q is formed by combining by addition T and P. R is formed by combining by addition T and Q. S is formed by combining by addition T and R.

B is formed by raising F to $\frac{4}{16}$ and combining by subtraction F and G. D is formed by combining by addition B and F. C is formed by combining by addition B and D. E is formed by combining by addition D and F. A is formed by raising B to $\frac{2}{16}$ and combining by subtraction B and C. The value of A, however, has a denominator which exceeds the given limit of 8 and hence must be omitted.

Checking for All Possible Intermediate Fractions within an Imposed Limit

The rules for determining whether there are additional intermediate fractions (whose terms are within an imposed limit) between two given fractions are as follows:

Rule 1. *Combine the fractions to be tested for intermediate fractions by addition, causing the denominator of the intermediate fraction to be equal to or slightly greater than the imposed limit.*

Rule 2. *Multiply the numerator of the first given fraction by the denominator of the newly formed fraction and the denominator of the first given fraction by the numerator of the newly formed fraction. Repeat this operation with the newly formed fraction and second fraction.*

Rule 3. *Observe the difference of these cross products. If the difference is* one *this indicates that there are no intermediate fractions between the two given fractions with a denominator less than the fraction having the greatest denominator. If the difference is* greater than one, *this indicates that there are other intermediate fractions with a denominator less than the fraction having the greatest denominator.*

Example a: In the foregoing problem (page 281) determine whether there are other intermediate fractions between the two given fractions C and D with a denominator less than 16.

Solution: Applying Rule 1, $3 + 2 = 5$ and $11 + 7 = 18$. Consequently the newly formed fraction is $\frac{5}{18}$. These fractions placed in their relative positions are as follows

$$\begin{array}{ccc} C & & D \\ \frac{3}{11} & \frac{5}{18} & \frac{2}{7} \end{array}$$

Applying Rule 2: $3 \times 18 = 54$ and $11 \times 5 = 55$. Likewise, $5 \times 7 = 35$ and $18 \times 2 = 36$.

Applying Rule 3: Since in each case the cross products differ by one, it indicates that there are no fractions between $\frac{3}{11}$ and $\frac{5}{18}$, or between $\frac{5}{18}$ and $\frac{2}{7}$ with a denominator less than 18. Also since the imposed limit is 16, it indicates that there are no fractions between $\frac{3}{11}$ and $\frac{2}{7}$ with a denominator less than 16. That is, $\frac{2}{7}$ is next in value to $\frac{3}{11}$ when working within the imposed limit.

Example b: Determine whether there are intermediate fractions between $\frac{43}{68}$ and $\frac{45}{71}$ with a denominator less than 100.

Solution: Applying Rule 1, $43 + 45 = 88$ and $68 + 71 = 139$. Consequently the newly formed fraction is $\frac{88}{139}$. These fractions placed in their relative positions are as follows:

$$\begin{array}{ccc} A & C & B \\ \frac{43}{68} & \frac{88}{139} & \frac{45}{71} \end{array}$$

Applying Rule 2: $43 \times 139 = 5977$ and $68 \times 88 = 5984$. Likewise $88 \times 71 = 6248$ and $139 \times 45 = 6255$.

Applying Rule 3: Since in each case the cross products differ by more than one, it indicates that there are other fractions between $\frac{43}{68}$ and $\frac{88}{139}$ and also between $\frac{88}{139}$ and $\frac{45}{71}$ with a denominator less than 139. The procedure for determining these fractions is as follows:

For convenience the fractions should be written with their numerators and denominators horizontally opposite each other and separated by a vertical line. Write the numerator

and denominator of the newly formed fraction first and then write under each, respectively, the numerator and denominator of the first fraction. Continue to add the terms of the first fraction to the combined sums until a fraction is produced which will reduce within the imposed limit. Thus:

88	139
43	68
131	207
43	68
174	275
43	68
217	343

Reducing this fraction ($\frac{217}{343}$) by dividing each term by 7 produces the fraction $\frac{31}{49}$. Combining C and B by the same process would produce the fraction $\frac{19}{30}$. Then combining D and E by addition would produce the fraction $\frac{50}{79}$.

These fractions again placed in their relative positions are as follows:

A	D	F	C	E	B
$\frac{43}{68}$	$\frac{31}{49}$	$\frac{50}{79}$	$\frac{88}{139}$	$\frac{19}{30}$	$\frac{45}{71}$

The fraction C may be crossed out because it is above the imposed limit and was added only to make the check possible. Applying the check to each pair of fractions in their successive order will indicate that there are no other fractions within this series with a denominator equal to or less than 100. The student should make this check.

Obtaining Intermediate Fractions Close to the Value of One of the Given Fractions

If it is desired to obtain fractions whose values are closer to one initial fraction than to the other, raise the terms of the fraction whose value is to be approached as high as possible according to the imposed limit, and combine this raised initial fraction and the other initial fraction by addition. Thus

beginning with the initial fractions, $\frac{1}{4}$ and $\frac{3}{8}$, let it be required to obtain an intermediate fraction whose value is close to $\frac{1}{4}$. If the terms of the fraction $\frac{1}{4}$ were raised by multiplying both the numerator and the denominator by 1000 and then combined by addition with the second initial fraction, the resulting fraction would be $\frac{1003}{4008}$. This value is obviously close to $\frac{1}{4}$. Likewise if the terms of the second initial fraction ($\frac{3}{8}$) were raised by 1000 and then combined by addition with the first initial fraction ($\frac{1}{4}$) the resulting fraction would be $\frac{3001}{8004}$. This value is very close to $\frac{3}{8}$.

Again consider the initial fractions $\frac{3}{5}$ and $\frac{5}{8}$. Let it be required to obtain an intermediate fraction whose value is close to $\frac{3}{5}$ and whose denominator shall not exceed 50. Raise $\frac{3}{5}$ to $\frac{24}{40}$ by multiplying both the numerator and the denominator by 8 and then combine the raised fraction with the second initial fraction. The resulting fraction will be $\frac{29}{48}$. The decimal equivalent of $\frac{29}{48}$ is .6042 while the decimal equivalent of $\frac{3}{5}$ is .6000 and of $\frac{5}{8}$ is .6250. It is therefore quite obvious that the value of the fraction $\frac{29}{48}$ is much closer to $\frac{3}{5}$ than to $\frac{5}{8}$. To find the successive intermediate fractions whose values are next greater or next less, according to the order of magnitude, and within an imposed limit, raise the terms of the initial fraction whose value is to be approached as high as possible so that after combining by addition with the other initial fraction, the resulting fraction will fall within the imposed limit. Next combine by subtraction continuously the intermediate fraction just obtained with the initial fraction (in its lowest terms) whose value was to be approached. Care should be taken that all the intermediate fractions are in their lowest terms before proceeding further. When the intermediate fraction can be reduced, consider this reduced fraction and the preceding intermediate fraction as two new initial fractions and proceed as before. To find the successive exterior fractions whose values are next greater or next lesser according to the order of magnitude, and within an imposed limit, raise the initial fraction to be approached to reasonable terms, and combine the other initial fraction by subtraction. Next combine the initial fraction to be approached and the

exterior fraction just obtained by addition and solve for their intermediate fractions as in the foregoing problem.

The following problem will illustrate the foregoing explanation, the procedure of which will be shown in a compact arrangement.

Example: Obtain the intermediate fractions of $\frac{3}{4}$ and $\frac{19}{25}$. The denominators of the intermediate fractions must not exceed 50. *For convenience the fractions will be written with their* numerators and denominators horizontally opposite each other and separated by a vertical line. Thus the work for this problem is as follows:

Solution:

3	4	
18	24	←$\frac{3}{4}$ raised by 6
19	25	
37	49	←Combining foregoing two by addition.
3	4	
34	45	←Combining foregoing two by subtraction
3	4	
31	41	←Continuing to combine result and $\frac{3}{4}$ by subtraction.
3	4	
28	37	
3	4	
25	33	
3	4	
22	29	
3	4	
19	25	←Finally the second initial fraction is reached.

PROBLEMS

1. How many intermediate fractions, whose denominators do not exceed 15, are there within the limiting values of $\frac{1}{4}$ and D?

2. Determine the numerator of the fourth intermediate fraction of E and $\frac{9}{16}$ considering E to be the first initial fraction, denominator not to exceed 23.

3. From the two fractions $\frac{3}{5}$ and R determine the numerator of the second exterior fraction, whose value is greater than R. Denominator not to exceed 16.

4. Determine the intermediate fraction nearest to the fraction G of the two initial fractions G and $\frac{11}{93}$, denominator not to exceed 300.

5. How many intermediate fractions, whose denominators do not exceed 33, are there within the limiting values of $\frac{3}{5}$ and H?

6. Determine the difference between the third and fourth intermediate fractions of K and $\frac{28}{45}$ considering K to be the first initial fraction. Denominator not to exceed 100. (Answer to be a decimal.)

7. From the two fractions S and $\frac{2}{3}$, determine the numerator of the second exterior fraction, whose value is less than S. Denominator not to exceed 16.

8. Determine the sixth intermediate fraction of M and $\frac{8}{19}$ considering M to be the first initial fraction, denominator not to exceed 100.

9. Determine the denominator of the third intermediate fraction of P and $\frac{33}{100}$ considering P to be the first initial fraction, denominator not to exceed 100.

10. From the two fractions $\frac{7}{15}$ and T determine the numerator of the third exterior fraction, whose value is greater than T. Denominator not to exceed 16.

VARIABLES

Prob.	Sym.	No. 1	No. 2	No 3	No. 4	No 5	No. 6
1	D	$\frac{3}{10}$	$\frac{4}{13}$	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{4}{11}$	$\frac{3}{8}$
2	E	$\frac{3}{5}$	$\frac{14}{23}$	$\frac{11}{18}$	$\frac{8}{13}$	$\frac{13}{21}$	$\frac{5}{8}$
3	R	$\frac{2}{3}$	$\frac{9}{13}$	$\frac{7}{10}$	$\frac{5}{7}$	$\frac{8}{11}$	$\frac{11}{15}$
4	G	$\frac{5}{42}$	$\frac{8}{67}$	$\frac{11}{92}$	$\frac{3}{25}$	$\frac{5}{42}$	$\frac{8}{67}$
5	H	$\frac{10}{17}$	$\frac{7}{12}$	$\frac{11}{19}$	$\frac{15}{26}$	$\frac{19}{33}$	$\frac{4}{7}$
6	K	$\frac{60}{97}$	$\frac{13}{21}$	$\frac{57}{92}$	$\frac{44}{71}$	$\frac{31}{50}$	$\frac{49}{79}$
7	S	$\frac{8}{5}$	$\frac{7}{12}$	$\frac{4}{7}$	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{6}{11}$
8	M	$\frac{5}{12}$	$\frac{38}{91}$	$\frac{33}{79}$	$\frac{28}{67}$	$\frac{23}{55}$	$\frac{41}{98}$
9	P	$\frac{24}{73}$	$\frac{25}{76}$	$\frac{26}{79}$	$\frac{27}{82}$	$\frac{28}{85}$	$\frac{29}{88}$
10	T	$\frac{6}{11}$	$\frac{5}{6}$	$\frac{9}{16}$	$\frac{4}{7}$	$\frac{7}{12}$	$\frac{3}{5}$

CONTINUED FRACTIONS

Definitions and Procedure for Obtaining Continued Fractions

Any fraction, such as $\frac{29}{67}$, may be converted into an equivalent fraction in the form of

$$\cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{2}}}}$$

which is called a **continued fraction.**

The following two steps are important in the formation of a continued fraction:

Step 1.—Without changing its value any common fraction can be replaced by a fraction whose numerator is 1 and whose denominator is the reciprocal of the fraction. Thus: $\frac{5}{8} = \frac{1}{\frac{8}{5}}$.

Step 2.—A mixed fraction, as $1\frac{3}{5}$, may be expressed in the form $1 + \frac{3}{5}$ without changing its value.

By Steps 1 and 2 the improper fraction $\frac{8}{5}$ can be replaced by the fraction $1 + \frac{1}{\frac{5}{3}}$, without changing the value of the improper fraction.

The following is an illustrative problem of the formation of continued fractions:

Illustrative Problem: Convert the fraction $\frac{29}{67}$ to a continued fraction.

Solution: By Step 1, $\frac{29}{67} = \frac{1}{\frac{67}{29}}$. Reduce $\frac{67}{29}$ to the mixed number $2\frac{9}{29}$ which by Step 2 may be written $2 + \frac{9}{29}$.

Replacing $\frac{67}{29}$ by $2 + \frac{9}{29}$ gives $\frac{29}{67} = \frac{1}{2_1 + \frac{9}{29}}$.

Indicate the successive quotients in the order obtained by subscripts. Thus the 2 in the foregoing equation is the first quotient obtained and is written 2_1.

Again, applying Step 1 to $\frac{9}{29}$, the continued fraction thus far is $\frac{29}{67} = \frac{1}{2_1 + \frac{1}{\frac{29}{9}}}$.

Changing $\frac{29}{9}$ to the mixed number $3\frac{2}{9}$ and applying Step 2,

$$\frac{29}{67} = \cfrac{1}{2_1 + \cfrac{1}{3_2 + \frac{2}{9}}}$$

Apply Steps 1 and 2 continuously until the mixed number has one for the numerator of its fractional part:

$$\frac{29}{67} = \cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{\frac{9}{2}}}} = \cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{4_3 + \cfrac{1}{2_4}}}}$$

The foregoing work may be put in a compact form by applying the same method as for finding the highest common factor (H.C.F.) of two numbers. The successive quotients are obtained as in finding the H.C.F. by continually dividing the larger numbers by the smaller until the last remainder becomes zero. The compact form of the foregoing problem is:

$$\begin{array}{r|r|l}
29 & 67 & 2_1 \\
27 & 58 & 3_2 \\
\hline
2 & 9 & 4_3 \\
2 & 8 & 2_4 \\
\hline
0 & 1 &
\end{array}$$

Convergents

In the continued fraction of $\frac{29}{67}$ or $\cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{4_3 + \cfrac{1}{2_4}}}}$ the fractions formed by neglecting everything to the right of the denominator indicated by the quotients in the order of 1, 2, 3, and 4 are $\frac{1}{2_1}$, $\cfrac{1}{2_1 + \cfrac{1}{3_2}}$, $\cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{4_3}}}$, $\cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{4_3 + \cfrac{1}{2_4}}}}$ which when simplified are

$$\frac{1}{2_1} = \frac{1}{2}, \qquad \frac{3_2}{3_2 \times 2_1 + 1} = \frac{3}{7}, \qquad \frac{4_3 \times 3_2 + 1}{4_3(3_2 \times 2_1 + 1) + 2_1} = \frac{13}{30}$$

$$\frac{2_4(4_3 \times 3_2 + 1) + 3_2}{2_4[4_3(3_2 \times 2_1 + 1) + 2_1] + 3_2 \times 2_1 + 1} = \frac{29}{67}.$$

These are called the first, second, third, and fourth convergents respectively.

The convergents taken in their consecutive order are alternately greater than and less than the complete value of the continued fraction, which may be shown for the foregoing problem as follows:

$$\cfrac{1}{2_1 + \cfrac{1}{3_2 + \cfrac{1}{4_3 + \cfrac{1}{2_4}}}}$$

Neglecting everything beyond the first quotient of the continued fraction will cause the denominator of the first convergent to be less than the entire denominator in the whole continued fraction, and therefore the value of the first convergent is greater than the complete value of the continued fraction.

Again, in the same continued fraction, neglecting everything beyond the second quotient will cause the value of the denominator to be less. Since the value of the second denominator is less, this in turn will cause the first denominator to be greater, thus causing the value of the second convergent to be less than the complete value of the continued fraction. Thus it may be seen that neglecting the quotients in their consecutive order produces successive convergents which are alternately greater than and less than the complete value of the continued fraction. The values of the successive convergents approach the complete value of the fraction and hence the value of the convergent closest to the last convergent is the nearest to the complete value of the continued fraction.

Theory for Obtaining Successive Convergents

Consider the simplified convergents already obtained for the fraction $\frac{29}{67}$:

$$\frac{1}{2_1}, \quad \frac{3_2}{3_2 \times 2_1 + 1}, \quad \frac{4_3 \times 3_2 + 1}{4_3(3_2 \times 2_1 + 1) + 2_1},$$
$$\frac{2_4(4_3 \times 3_2 + 1) + 3_2}{2_4[4_3(3_2 \times 2_1 + 1) + 2_1] + 3_2 \times 2_1 + 1}.$$

By inspection, it may be noticed that the numerator of the third convergent is composed of the product of the third quotient and the numerator of the second convergent, plus the numerator of the first convergent, and similarly the denominator of the third convergent is composed of the product of the third quotient and the denominator of the second convergent, plus the denominator of the first convergent. It may also be seen by inspection that the numerator and denominator of

the fourth convergent can be obtained in like manner from the fourth quotient and the numerators and denominators of the two preceding convergents. In order to obtain the first two convergents by this procedure, two initial convergents must be used. Since the convergents are alternately greater and less than the original fraction the two initial convergents may be infinity (written $\frac{1}{0}$ in this case) and zero (written $\frac{0}{1}$ in this case).

If the fraction to be considered is an improper fraction, the initial convergents to be used are $\dfrac{1}{0}$ and $\dfrac{r}{1}$ where r is the first quotient obtained.

Rules for Obtaining Successive Convergents

Rule 1.—*Compute the successive quotients by the compact-form method.*

Rule 2.—*Draw a diagram as shown in Fig.* 191 *with sufficient divisions to take care of the initial convergents and the successive quotients.*

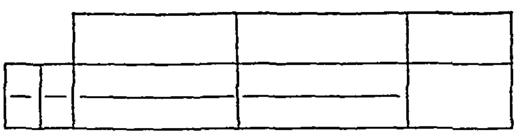


FIG. 191.

Rule 3.—*Write the initial convergents in the two spaces at the left and the successive quotients in the upper row of spaces, as in Fig.* 192.

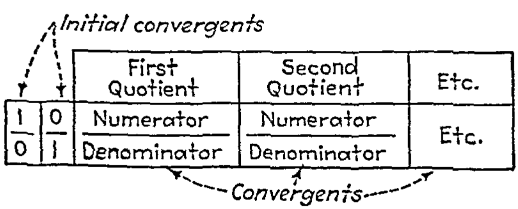


FIG. 192.

Rule 4.—*Starting with the first quotient and continuing with the successive quotients to the right, the* $\left(\begin{matrix}\textit{numerator}\\ \textit{denominator}\end{matrix}\right)$ *of the suc-*

cessive convergents are obtained by multiplying the quotient by the $\left(\frac{\text{numerator}}{\text{denominator}}\right)$ *of the first preceding convergent (which is in the first division to the left) and adding the* $\left(\frac{\text{numerator}}{\text{denominator}}\right)$ *of the second preceding convergent (which is in the second division to the left).*

Example a: Find the successive convergents for the fraction $\frac{5855}{8477}$.

Solution:

Rule 1.—Find the successive quotients, by continually dividing the larger number by the smaller number until the last remainder is zero. The work in the compact form is shown below:

5855	8477	1	First quotient
5244	5855	2	Second quotient
611	2622	4	Third quotient
534	2444	3	Fourth quotient
77	178	2	Fifth quotient
72	154	3	Sixth quotient
5	24	4	Seventh quotient
4	20	1	Eighth quotient
1	4	4	Ninth quotient
	4		
	0		Last remainder

Rules 2 and 3.—Draw frame and insert quotients and initial convergents.

		1	2	4	3	2	3	4	1	4
$\frac{1}{0}$	$\frac{0}{1}$									

Rule 4.—The successive convergents are formed according to Rule 4 and are tabulated together with their decimal values in the frame on the following page:

		A	B	C	D	E	F	G	H	I
		1	2	4	3	2	3	4	1	4
$\frac{1}{0}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{3}$	$\frac{9}{13}$	$\frac{29}{42}$	$\frac{67}{97}$	$\frac{230}{333}$	$\frac{987}{1429}$	$\frac{1217}{1762}$	$\frac{5855}{8477}$
Greater infinity	Less .00000000	Greater 1.00000000	Less .66666667	Greater .69230769	Less .69047619	Greater .69072164	Less .69069069	Greater .69069279	Less .69069239	Original .69069246

Numerator of	Denominator of
$A = 1 \times 0 + 1 = 1.$	$A = 1 \times 1 + 0 = 1.$
$B = 2 \times 1 + 0 = 2.$	$B = 2 \times 1 + 1 = 3.$
$C = 4 \times 2 + 1 = 9.$	$C = 4 \times 3 + 1 = 13.$
$D = 3 \times 9 + 2 = 29.$	$D = 3 \times 13 + 3 = 42.$
$E = 2 \times 29 + 9 = 67.$	$E = 2 \times 42 + 13 = 97.$
$F = 3 \times 67 + 29 = 230.$	$F = 3 \times 97 + 42 = 333.$
$G = 4 \times 230 + 67 = 987.$	$G = 4 \times 333 + 97 = 1429.$
$H = 1 \times 987 + 230 = 1217.$	$H = 1 \times 1429 + 333 = 1762.$
$I = 4 \times 1217 + 987 = 5855.$	$I = 4 \times 1762 + 1429 = 8477.$

Note: The difference between the values of any two consecutive convergents is always equal to the reciprocal of the product of the two denominators. Thus in the results of Example *a*, the difference between D and E is $.69072164 - .69047619 = .00024545$.

$$\frac{1}{42 \times 97} = \frac{1}{4074} = .00024545.$$

Hence in checking the nearness of any convergent to the exact value of the given fraction, the deviation of any convergent from the exact value will be less than the difference between this convergent and the preceding convergent, which is easily obtained as above.

Example b: Find the successive convergents for the fraction $\frac{147}{62}$.

Solution:

Rule 1: The quotients by the compact form are obtained as follows:

147	62	2_0
124	46	2_1
23	16	1_2
16	14	2_3
7	2	3_4
6	2	2_5
1	0	

Note: Since $\frac{147}{62}$ is an improper fraction, the first quotient obtained, 2_0, is the numerator of the second initial convergent and the next quotient, 2_1, is considered as being the first of the series of successive quotients.

Applying Rules 2, 3, and 4.

		2	1	2	3	2
$\frac{1}{0}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{7}{3}$	$\frac{19}{8}$	$\frac{64}{27}$	$\frac{147}{62}$

If it is required to obtain the continued fraction of a decimal number, the decimal number must be changed to a common fraction.

Example a: .68767 is $\frac{68767}{100000}$ which is obtained by using the figures in the decimal number as the numerator and as many ciphers as there are figures to the right of the decimal point preceded by 1 for the denominator.

Example b: 2.539 $= \frac{2539}{1000}$.

In case a continued fraction of the square root of a number is desired, extract the square root of the number and proceed as above.

PROBLEMS

1. What convergent should be placed in the block corresponding to X?

		2	4	1	2	4	3	5	1	2	2
$\frac{1}{0}$	$\frac{0}{1}$	A	B	C	D	E	F	G	H	J	K

2. What improper fraction is a very close approximation of $\sqrt{D}$? Numerator not to exceed 100.

3. What improper fraction is a very close approximation of $\frac{E}{4091}$? Numerator not to exceed 100.

4. What common fraction is a very close approximation of G? Denominator not to exceed 100.

5. What convergent should be placed in the block corresponding to X?

		1	3	2	4	1	2	1	2	2	1	2	3	2
$\frac{1}{0}$	$\frac{0}{1}$	A	B	C	D	E	F	G	H	J	K	L	M	N

6. What is the value of the convergent just found in the preceding problem, correct to five places?
(Notice the difference between this convergent and the one preceding.)

7. What common fraction is a very close approximation of $\frac{100}{H}$? Denominator not to exceed 600.

8. What improper fraction is a very close approximation of J? Numerator not to exceed 800.

9. What common fraction is a very close approximation of K? Denominator not to exceed 10000.

10. What convergent should be placed in the block corresponding to X?

		2	5	3	1	1	1	3	6	1	3	2
$\frac{1}{0}$	$\frac{0}{1}$	A	B	C	D	E	F	G	H	J	K	L

VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	X	J	H	G	F	E	D
2	D	8	7	6	5	3	2
3	E	8543	7985	7105	6987	5973	5678
4	G	.9326	.8129	.7879	.6785	.5987	.5673
5	X	G	H	J	K	L	M
6	X	G	H	J	K	L	M
7	H	895.372	751.468	699.729	687.213	587.873	468.751
8	J	4.28767	3.89763	3.28757	2.89763	2.13957	1.87693
9	K	.87879	.75373	.67239	.58787	.49873	.43589
10	X	F	G	H	J	K	L

SOLVING FOR FACTORABLE NUMBERS FOR USE IN COMPOUND GEARING

In order to teach the theory of rearranging gears into compound order, it was necessary to select numbers that would factor; but in practice it seldom occurs that these numbers will factor. This difficulty is overcome through the aid of continued fractions and combining fractions which enables one to resolve into factors any ratio with but a slight variation

from the original value. In continued fractions the convergents are alternately greater or less than the original, which in turn produces leads that are slightly greater or less. One should determine whether the lead to be cut is to be greater or less than the original. If the ratio of simple gears connecting the spindle and lead screw be taken from the greater column, the lead produced will be greater; but if taken from the lesser column the lead cut will be less. An illustrative problem will be used in explaining the details of solving for factorable numbers:

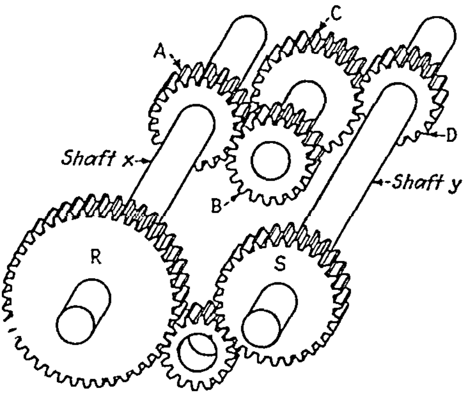


FIG. 193.

Illustrative Problem: Determine the number of teeth in A, B, C, and D gears if R and S have 14.954 and 5.805 teeth, respectively.

Solution: It must be remembered that these decimal numbers of teeth are derived from the revolutions of the shafts. This ratio may be converted into the following successive convergents:

		A	B	C	D	E	F	G	H	I	J	K
		1	1	2	1	3	1	2	3	2	1	5
$\frac{1}{0}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{5}{2}$	$\frac{13}{5}$	$\frac{18}{7}$	$\frac{67}{26}$	$\frac{85}{33}$	$\frac{237}{92}$	$\frac{796}{309}$	$\frac{1829}{710}$	$\frac{2625}{1019}$	$\frac{14954}{5805}$
G	L	G	L	G	L	G	L	G	L	G	L	O

First let us consider the situation in which the lead is to be slightly greater. This is equivalent to saying that the Y shaft must revolve a little faster. Since the Y shaft revolves faster, the fraction representing the ratio must be greater than the original. It happens in this problem that the I convergent which has a greater value than the original is factorable. However, to show the general procedure, the problem will be continued to find other factorable numbers.

Since the numbers of teeth are usually within the range of 20 to 100, no convergent having a numerator or denominator greater than 100×100 or 10,000 should be used. Hence in this problem, the convergent K cannot be used, and the convergent J is used as the starting point.

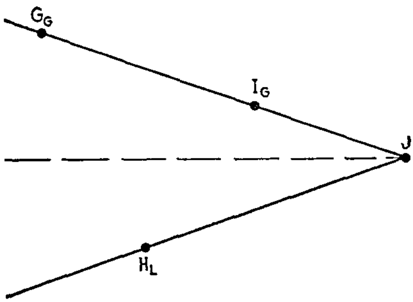


FIG. 194.

Since the convergents of the continued fractions are alternately greater and less than the original value, they may be represented by diagrams as shown in Fig. 194 where I_G, H_L, G_G are the next convergents greater and less than the starting convergent J.

By means of combining fractions (by both addition and subtraction), obtain all the intermediate fractions from J to I_G until one is found which is factorable. In order to form fractions as close to J as possible, raise J to as high terms as the limit of 10,000 allows and combine with I_G by addition, and then continue to combine by subtraction with the number that has been raised.

Diagrammatic Form for Continued Fractions

Since the convergents of a continued fraction are alternately greater than and less than the original number, a closer study of the convergents in connection with the theory of combining fractions may be made by the aid of a diagrammatic form as shown in Figs. 195 and 196. The diagrammatic *method briefly referred to in Fig. 194 may be more clearly* understood by the following explanation.

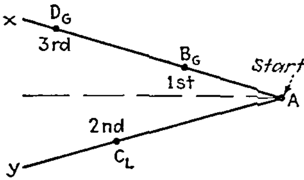


FIG. 195.

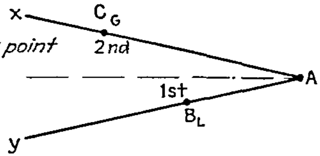


FIG. 196.

In Figs. 195 and 196 let the different positions on the inclined line AX, which represent the convergents and fractions formed by combining the convergents, be the values *greater* than the convergent selected as the starting point. Also let the different positions on the declined line AY, which represent the convergents and fractions formed by combining the convergents, be the values *less* than the convergent selected as the starting point. The starting point is the convergent just within the limit imposed. Assume that it is required to get other fractions by the theory of combining convergents whose values are greater than the value of the convergent *used as the starting point A.*

The figure represents the case where the next convergent to the left (B_G) is greater than the convergent A, selected as the starting point. If for this case the required values are to be greater than the original number, the procedure is to raise A to higher terms and combine with B_G by addition. Then continue to combine this result with A by subtraction, thus giving values between A and B_G. To get all the possible values between A and B_G, combine by addition the values already found, keeping within the imposed limit. To prevent

duplicating values thus obtained and also to give the greatest possible number of values between B_G and D_G, raise B_G and combine C_L by subtraction, and continue to combine the remainders by subtraction with B_G until either D_G or a negative result is reached. If a negative result is reached, discard it, raise the last positive value obtained, again combine with C_L by subtraction, and continue to combine the remainder by subtraction with this last positive value (not raised).

Again suppose it is required to find values greater than the convergent A in Fig. 196, where the first convergent to the left (B_L) is less in value. The values from A to C_G are obtained in the same way in which the values for the foregoing case (Fig. 195) were obtained from B_G to D_G by raising A to higher terms, combining with B_L by subtraction and continuing to combine the remainder with A by subtraction, etc., until C_G is reached. Then similarly raise C_G to higher terms, combine by subtraction with D_L, and continue to combine the remainder by subtraction with C_G, etc.

To obtain values less than the convergent which was selected for the starting point, as in the case shown in Fig. 195, carry out the work in the same manner in which the greater values were obtained for the case shown in Fig. 196 and *vice versa.*

Before continuing the problem of getting a fraction, close to the value of $\frac{14,954}{5805}$, which will factor, the ideas just discussed will be summarized in the form of general rules which will then be applied to the completion of this problem.

General Rules for Obtaining Factorable Numbers

Rule 1.—*Raise the convergent, closest to the original number, as high as possible so that after combining as in Rule* 2, *the result will be within the limit of* 10,000.

Rule 2.—*Combine this raised convergent with the next convergent to the left of the convergent which has been raised, by addition or subtraction depending upon whether a greater or a lesser value than the original fraction is desired.*

Rule 3.—*Continue to combine by subtraction, as many times as possible, the fraction which has been raised, with the result of Rule 2. If the remainder is not a convergent, raise it as high as possible, combine by subtraction with the convergent whose denominator is the next number less than the denominator of the remainder, and continue to combine by subtraction this result with the remainder (not raised). If necessary, continue the process of this rule until the next convergent is reached.*

Rule 4.—*Combine by addition any two succeeding values obtained in Rule 2 or Rule 3 which will give results that will not exceed the imposed limit.*

In the problem stated on page 296 the successive convergents have already been obtained and are repeated below for convenient reference.

		A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*
		1	1	2	1	3	1	2	3	2	1	5
$\frac{1}{0}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{5}{2}$	$\frac{13}{5}$	$\frac{18}{7}$	$\frac{67}{26}$	$\frac{85}{33}$	$\frac{237}{92}$	$\frac{796}{309}$	$\frac{1829}{710}$	$\frac{2625}{1019}$	$\frac{14954}{5805}$
G	*L*	*G*	*L*	*G*	*L*	*G*	*L*	*G*	*L*	*G*	*L*	*O*

The numerators and denominators of the convergents and fractions given below are placed side by side in order that they may be combined by subtraction conveniently, and in order that it may be seen at a glance which fractions may be combined by addition.

Applying the first three rules to the foregoing convergents to get values greater than the original value, the steps are as follows:

(1)	9704	3767	← Multiply the *J* convergent by 3 and combine by addition the *I* convergent.
	2625	1019	
(2)	7079	2748	← Continue to combine by subtraction the *J* convergent.
	2625	1019	
(3)	4454	1729	
	2625	1019	
(4)	1829	710	

(5)	8349	3241	← Multiply the *I* convergent by 5 and combine by subtraction the *H* convergent.
	1829	710	
(6)	6520	2531	← Continue to combine by subtraction the *I* convergent.
	1829	710	
(7)	4691	1821	
	1829	710	
(8)	2862	1111	
	1829	710	
(9)	1033	401	
(10)	9534	3701	← Since the foregoing remainder is not a convergent. raise by 10, and combine by subtraction the *H* convergent.
	1033	401	
(11)	8501	3300	← Continue to combine by subtraction the remainder (not raised).
	1033	401	
(12)	7468	2899	
(13)	6435	2498	← To save space, only the results will be tabulated for the following steps.
(14)	5402	2097	
(15)	4369	1696	
(16)	3336	1295	
(17)	2303	894	
(18)	1270	493	
(19)	237	92	← This fraction is the next convergent and the foregoing routine should be repeated if necessary.

Applying Rule 4 to the foregoing values, gives many intermediate fractions, indicated by letters, which may be inserted as shown below:

(1)	9704	3767	$\frac{14.954}{5.805} = 2.5760551$ = original value.
(2)	7079	2748	
(3)	4454	1729	
(*a*)	6283	2439	
(*b*)	8112	3149	
(*c*)	9941	4859	
(4)	1829	710	← $\frac{1829}{710} = \frac{31 \times 59}{71 \times 10} = \frac{62 \times 59}{71 \times 20} = \frac{AC}{BD}$, Where *A*, *B*, *C*, and *D* are the numbers of teeth in the gears.
(5)	8349	3241	
(6)	6520	2531	
(7)	4691	1821	$\frac{1829}{710} = 2.5760563$. Variation from the original value is .0000012.
(*a*)	7553	2932	

(8)	2862	1111	
(*a*)	9619	3734	
(*b*)	6757	2623	
(*c*)	3895	1512	← $\dfrac{3895}{1512} = \dfrac{95 \times 41}{24 \times 63} = \dfrac{AC}{BD}$.
(*d*)	8823	3425	$\dfrac{3895}{1512} = 2.5760582$. Variation from the original value is .0000031.
(*e*)	4928	1913	
(*f*)	5961	2314	
(*g*)	6994	2715	
(*h*)	8027	3116	
(*i*)	9060	3517	
(9)	1033	401	
(10)	9534	3710	
(11)	8501	3300	
(12)	7468	2899	
(13)	6435	2498	
(14)	5402	2097	
(15)	4369	1696	
(16)	3336	1295	There are approximately 100 intermediate fractions which could be inserted between these fractions. If the foregoing numbers of teeth in gears are not suitable, find these intermediate fractions and examine for further factorable numbers.
(17)	2303	894	
(18)	1270	493	
(19)	237	92	

In order to find factorable numbers less than the original number, which will cause the lead to be slightly less (which is equivalent to saying that the Y shaft must revolve a little slower), the work is carried out as follows.

Applying the first three rules:

(1)	2625	1019	← J convergent.
(2)	8671	3366	← Multiply the J convergent by 4, and combine by subtraction the I convergent.
(3)	6046	2347	
(4)	3421	1328	Continue to combine by subtraction the J convergent.
(5)	796	309	← H convergent.
(6)	9315	3617	← Multiply the H convergent by 12 and combine by subtraction the G convergent.
(7)	8519	3307	
(8)	7723	2998	Continue to combine by subtraction the H convergent.
(9)	6927	2689	
(10)	6131	2380	

(11)	5335	2071	
(12)	4539	1762	
(13)	3743	1453	
(14)	2947	1144	
(15)	2151	835	
(16)	1355	526	This number is factorable but the work will be continued to show the general procedure.
(17)	559	217	
(18)	9825	3814 ←	Using Rule 3, multiply the foregoing remainder by 18 and combine by subtraction with G.
(19)	9266	3597	
(20)	8707	3380	Continue to combine by subtraction the remainder (not raised).
(21)	8148	3163	
(22)	7589	2946	
(23)	7030	2728	
(24)	6471	2512	
(25)	5912	2295	
(26)	5353	2078	
(27)	4794	1861	
(28)	4235	1644	
(29)	3676	1427	
(30)	3117	1210	
(31)	2558	993	
(32)	1999	776	
(33)	1440	559	
(34)	881	342	
(35)	322	125	
(36)	9745	3783 ←	Again applying Rule 3, multiply the foregoing remainder by 31 and combine by subtraction with G.
(37)	9383	3668	Continue to combine by subtraction the remainder (not raised). Carrying out this work thirty times (of which only the first is given here) will lead to the F convergent.

Applying Rule 4 to the foregoing values gives many intermediate fractions, indicated by letters, which may be inserted as shown in the following:

(1)	2625	1019	(*i*)	8926	3465
(2)	8671	3366	(16)	1355	526
(3)	6046	2347	(*a*)	8689	3373
(*a*)	9467	3675	(*b*)	7334	2847
(4)	3421	1328	(*c*)	5979	2321
(*a*)	7638	2965	(*d*)	4624	1795
(*b*)	4217	1637	(*e*)	7893	3064

(8)	2862	1111	
(a)	9619	3734	
(b)	6757	2623	
(c)	3895	1512	$\leftarrow \frac{3895}{1512} = \frac{95 \times 41}{24 \times 63} = \frac{AC}{BD}$.
(d)	8823	3425	$\frac{3895}{1512} = 2.5760582$. Variation from the original value is .0000031.
(e)	4928	1913	
(f)	5961	2314	
(g)	6994	2715	
(h)	8027	3116	
(i)	9060	3517	
(9)	1033	401	
(10)	9534	3710	
(11)	8501	3300	
(12)	7468	2899	
(13)	6435	2498	
(14)	5402	2097	
(15)	4369	1696	
(16)	3336	1295	There are approximately 100 intermediate fractions which could be inserted between these fractions. If the foregoing numbers of teeth in gears are not suitable, find these intermediate fractions and examine for further factorable numbers.
(17)	2303	894	
(18)	1270	493	
(19)	237	92	

In order to find factorable numbers less than the original number, which will cause the lead to be slightly less (which is equivalent to saying that the Y shaft must revolve a little slower), the work is carried out as follows.

Applying the first three rules:

(1)	2625	1019	← *J convergent.*
(2)	8671	3366	← Multiply the J convergent by 4, and combine by
(3)	6046	2347	subtraction the I convergent.
(4)	3421	1328	Continue to combine by subtraction the J convergent.
(5)	796	309	← H convergent.
(6)	9315	3617	← Multiply the H convergent by 12 and combine by
(7)	8519	3307	subtraction the G convergent.
(8)	7723	2998	Continue to combine by subtraction the H convergent.
(9)	6927	2689	
(10)	6131	2380	

(11)	5335	2071	
(12)	4539	1762	
(13)	3743	1453	
(14)	2947	1144	
(15)	2151	835	
(16)	1355	526	This number is factorable but the work will be continued to show the general procedure.
(17)	559	217	
(18)	9825	3814	Using Rule 3, multiply the foregoing remainder by 18 and combine by subtraction with G.
(19)	9266	3597	
(20)	8707	3380	Continue to combine by subtraction the remainder (not raised).
(21)	8148	3163	
(22)	7589	2946	
(23)	7030	2728	
(24)	6471	2512	
(25)	5912	2295	
(26)	5353	2078	
(27)	4794	1861	
(28)	4235	1644	
(29)	3676	1427	
(30)	3117	1210	
(31)	2558	993	
(32)	1999	776	
(33)	1440	559	
(34)	881	342	
(35)	322	125	
(36)	9745	3783	Again applying Rule 3, multiply the foregoing remainder by 31 and combine by subtraction with G.
(37)	9383	3668	Continue to combine by subtraction the remainder (not raised). Carrying out this work thirty times (of which only the first is given here) will lead to the F convergent.

Applying Rule 4 to the foregoing values gives many intermediate fractions, indicated by letters, which may be inserted as shown in the following:

(1)	2625	1019	(*i*)	8926	3465
(2)	8671	3366	(16)	1355	526
(3)	6046	2347	(*a*)	8689	3373
(*a*)	9467	3675	(*b*)	7334	2847
(4)	3421	1328	(*c*)	5979	2321
(*a*)	7638	2965	(*d*)	4624	1795
(*b*)	4217	1637	(*e*)	7893	3064

The following problems refer to Fig 197.

6. Determine the closest set of gears that can be obtained for A, B, C, and D, the Y shaft being permitted to revolve a little slower, if the numbers of revolutions of the shafts X and Y are M and 12.6214, respectively.

7. Determine the numbers of teeth in gears A, B, C, and D, if R and S have 6.573 and H teeth, respectively.

8. Determine the numbers of teeth in gears A, B, C, and D, if R and S have J and 4.371 teeth, respectively.

VARIABLES

No.	Sym.	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6
6	M	10 9375	9 9875	9.3786	8.9879	8.2985	7.8953
7	H	8 767	9 135	9.269	9.563	9.728	9.819
8	J	2 356	2 567	2 756	2 931	3.129	3.265

CONTINUED FRACTIONS APPLIED TO CUTTING LEADS ON A LATHE

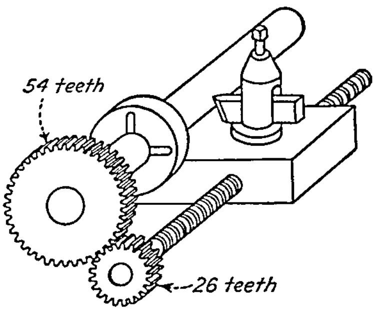


FIG. 198.

In order to place the proper gears on a lathe, when a certain lead is being cut, the following steps are necessary: Determine the number of revolutions the lead screw makes while the spindle makes 1 revolution. Consider the number of revolutions the lead screw makes to be equal to the number of teeth in the gear on spindle, and the number of revolutions the spindle makes to be equal to the number of teeth in the gear on leadscrew.

The number of revolutions that the lead screw makes when examined in gives 1 revolution is equal to the required lead
of the thread on lead screw.

Example: Determine the compound gears on a lathe in order to cut a lead equal to .3461 in.

Solution: The lead screw has six threads per inch; consequently the lead on lead screw is $\frac{1}{6}$ in. The number of revolutions the lead screw makes when the shaft makes 1 revolution is equal to $.3461 \div \frac{1}{6}$ or 2.0766 revolutions. Consider this number of revolutions (2.0766) to be the number of teeth in the gear on spindle, and the number of teeth in gear on lead screw to be equal to 1. In order to eliminate the decimal point, multiply both the number of teeth in the gear on spindle and the number of teeth in the gear on the lead screw by 10,000, which gives

$$\frac{2.0766}{1} \times \frac{10{,}000}{10{,}000} = \frac{20{,}766}{10{,}000}.$$

Assume that the number of teeth in the gears must be within the limits of 20 and 100. Since the numerator and denomi-

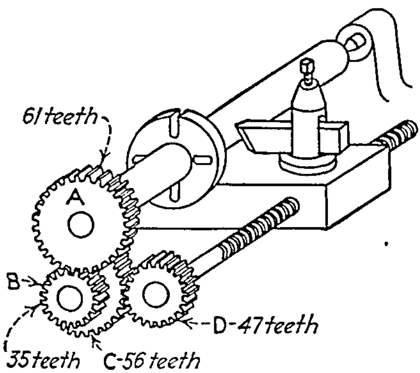


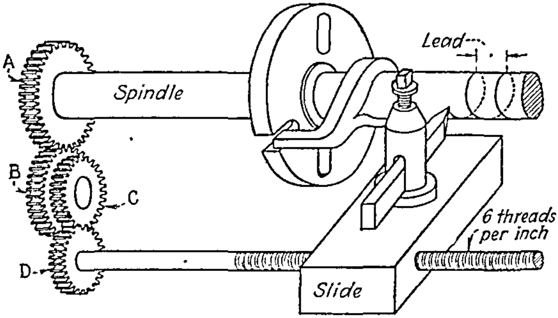
FIG. 199.

nator in the foregoing fraction are over 100, it becomes necessary to solve for a more convenient fraction by the aid of continued fractions. Obtaining the convergents for the foregoing fraction by continued fractions gives for simple gears $\frac{27}{13}$ or $\frac{54}{26}$. If the 54-tooth gear were placed on the spindle and the 26-tooth gear on the lead screw, the distance that the tool on the carriage of the lathe would move when the spindle makes 1 revolution would be $\frac{54}{26}$ times $\frac{1}{6}$ or .34615 in. The variation

between the required lead and the lead cut by the two gears having 54 and 26 teeth is thus .00005 in. This variation on a short thread screw might be permissible; but if the thread screw were 34 in. long, the variation from one end to the other would be .005 in. and necessitates rearranging the simple gears having 20,766 and 10,000 teeth into compound order.

Since a compound arrangement is made up of four or more gears, a convergent may be chosen whose numerator and denominator are considerably larger. In this case the next convergent is $\frac{488}{235}$ and can be used very conveniently if the lead must be more accurate. In fact, using the convergent nearest to the original fraction will give the best result in the lead. The factors of the fraction $\frac{488}{235}$ are $(61 \times 8) \div (47 \times 5)$ which raised to higher terms are $(61 \times 56) \div (47 \times 35)$. Placing these gears into compound order gives $A = 61$, $B = 35$, $C = 56$, and $D = 47$. The lead that these gears will cut is $\frac{488}{235} \times \frac{1}{6}$ or .3460993. The variation in this case is equal to .0000007.

PROBLEMS



Determine the numbers of teeth in the gears A, B, C, and D, in order cut the following leads.

1. L = lead to be cut Lead screw has 8 threads per inch.
2. K = lead to be cut. Lead screw has 6 threads per inch.
3. M = lead to be cut. Lead screw has 4 threads per inch.
4. N = lead to be cut. Lead screw has 8 threads per inch.
5. P = lead to be cut. Lead screw has 6 threads per inch.
6. Q = lead to be cut. Lead screw has 4 threads per inch.

VARIABLES

Prob	Sym.	No. 1	No. 2	No 3	No. 4	No. 5	No 6
1	L	.7854	.7286	.6985	.6352	.5916	.5432
2	K	.3768	.4697	.5683	.6782	.7895	.8379
3	M	.2753	.3259	.3578	.3753	.3941	.4297
4	N	.6354	.6579	.6783	.6927	.7013	.7255
5	P	.4379	.4563	.4791	.4787	.4965	.5167
6	Q	.1356	.1567	.1789	.1935	.2251	.2457

CONTINUED FRACTIONS APPLIED TO LEADS ON A MILLING MACHINE

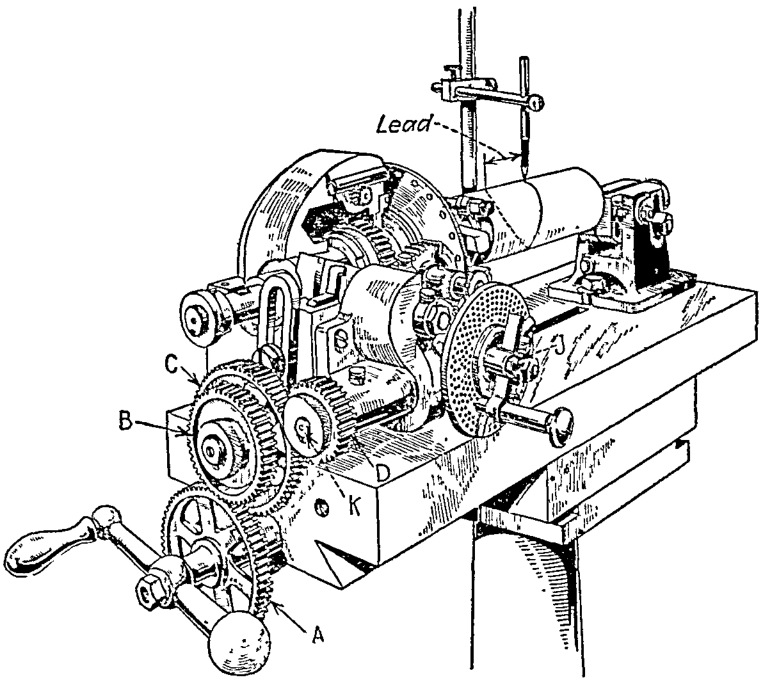


FIG. 200.

The cutaway dividing head which is mounted on a special demonstration table connected by change (compound) gears is shown in Fig. 200 to give the student a better conception of the working parts, which will be briefly discussed. Mounted

on the J shaft is a worm which is in mesh with the worm wheel on the spindle. The ratio of these two units is 40 to 1 which means that the crank on the J shaft makes 40 revolutions to each revolution of the spindle. The K shaft is connected with the J shaft by means of spiral gears which have equal numbers of teeth. Since they each have the same number of teeth, the K shaft will also make 40 revolutions to each revolution of the spindle. The revolutions of the K shaft depend upon the ratio of the worm and worm wheel and in most cases this ratio is either 40 to 1 or 60 to 1.

The lead screw indicated is used to move the table in a longitudinal direction. The distance that the table moves for each revolution of the lead screw depends upon the number of threads per inch on this screw. The lead screws in most milling machines have four threads per inch which means that for each revolution of the lead screw the table will move $\frac{1}{4}$ in. Since the lead of a single-threaded screw is the reciprocal of the number of threads per inch, then the lead of the lead screw in this case is $\frac{1}{4}$ in. The number of revolutions that the lead screw makes when the table moves a distance equal to the lead to be cut is equal to the lead to be cut divided by the lead of the lead screw. Since the lead of a single-threaded screw is equal to 1 divided by the number of threads per inch, it is evident that the number of revolutions of the lead screw is equal to the lead to be cut multiplied by the number of threads on the lead screw. It must be borne in mind that the spindle makes 1 revolution when the table moves a distance equal to the lead to be cut. In using a dividing head whose worm and worm-wheel ratio is 40 to 1, the ratio of the number of revolutions of the lead screw to the number of revolutions of the K shaft is the product of the lead to be cut and the number of threads per inch on the lead screw to 40.

In Fig. 201, the dividing head is shown mounted on the table on a milling machine, and its gears and shafts are labeled to correspond with Fig. 200.

To determine the proper gears to be used on the dividing head of a milling machine when cutting a certain lead, the

procedure is as follows: Determine the number of revolutions that the lead screw makes while the spindle makes 1 revolution by multiplying the lead to be cut by the number of

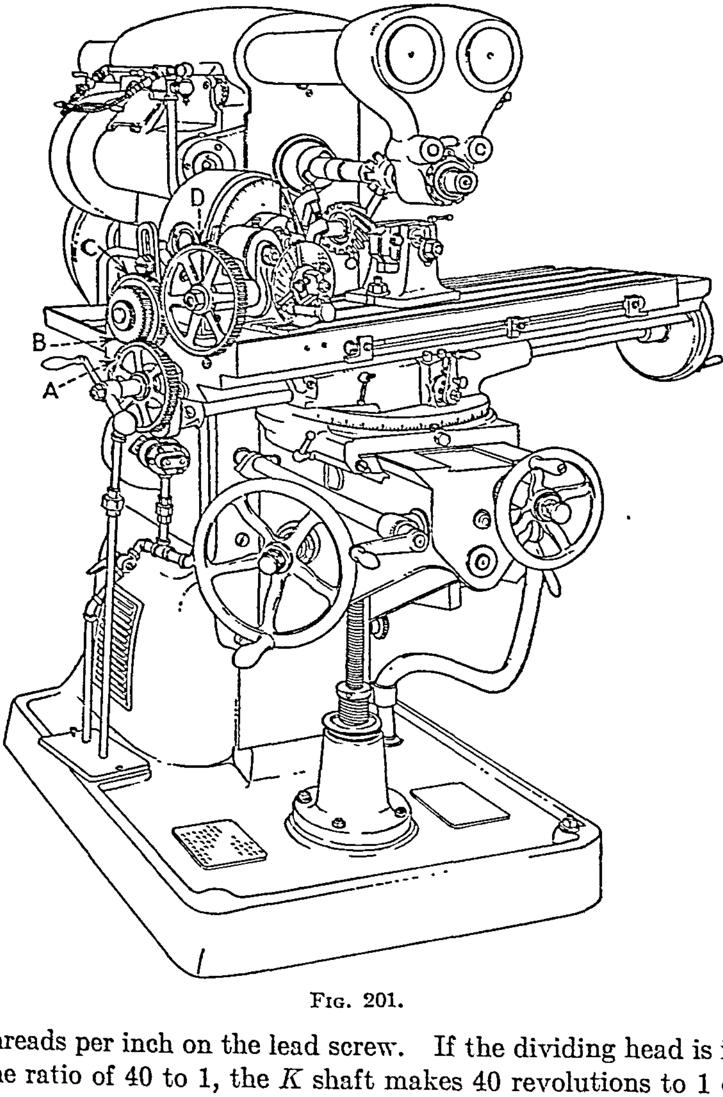


FIG. 201.

threads per inch on the lead screw. If the dividing head is in the ratio of 40 to 1, the K shaft makes 40 revolutions to 1 of the spindle. Consider the number of revolutions of the lead screw to be the number of teeth in the simple gear on the K

shaft and the number of revolutions of the K shaft (40) to be the number of teeth in the simple gear on the lead screw. In order to formulate this procedure, the following notation will be used:

L = lead to be cut.
n = number of threads per inch on lead screw.
R = number of revolutions K shaft makes to one of spindle.

A, B, C, and D are numbers of teeth in compound gears.

$$\frac{AC}{BD} = \frac{R}{Ln}.$$

Example: Let it be required to cut a lead equal to 12.240925 on a milling machine having four threads per inch, using a dividing head having a ratio of 40 to 1.

Solution: Applying the formula:

$$\frac{AC}{BD} = \frac{R}{Ln} = \frac{40}{12.240925 \times 4} = \frac{40}{48.9637}.$$

Multiplying numerator and denominator by 10,000 to eliminate the decimal point gives

$$\frac{AC}{BD} = \frac{400000}{489637}.$$

This ratio must be simplified by the use of continued fractions. The work for obtaining the quotients is:

489637	400000	1
400000	358548	4
89637	41452	2
82904	40398	6
6733	1054	6
6324	818	2
409	236	1
236	173	1
173	63	2
126	47	1
47	16	2
32	15	1
15	1	15
15		
0		

The convergents are obtained from these quotients in the usual manner and are as follows:

		4	2	6	6	2	1	1	2	1	2	1	15
$\frac{1}{0}$	$\frac{1}{1}$	$\frac{5}{4}$	$\frac{11}{9}$	$\frac{71}{58}$	$\frac{437}{357}$	$\frac{945}{772}$	$\frac{1382}{1129}$	$\frac{2327}{1901}$	$\frac{6036}{4931}$	$\frac{8363}{6832}$	$\frac{22762}{18595}$	$\frac{31125}{25427}$	$\frac{489637}{400000}$

Starting with the convergent $\frac{8363}{6832}$ and solving for the series of greater values by applying the four rules gives, at about the fiftieth step, the fraction $\frac{1961}{1602}$ which is factorable, the factors being

$$\frac{1602}{1961} = \frac{37 \times 53}{18 \times 89} = \frac{74 \times 53}{36 \times 89} = \frac{AC}{BD}.$$

In order to check the lead that will be obtained with these gears, solve for L in the original formula.

$$L = \frac{RBD}{nAC} = \frac{40 \times 1961}{4 \times 1602} = 12.240948$$

which is a variation of .000023 from the required lead.

Starting with the same convergent $\frac{8363}{6832}$ and solving for the series of lesser values gives a factorable result in the fourth step, the fraction being $\frac{8800}{7189} = \frac{88 \times 100}{79 \times 91} = \frac{BD}{AC}$.

Checking:

$$L = \frac{RBD}{nAC} = \frac{40 \times 8800}{4 \times 7189} = 12.240923$$

which is a variation of only .000002 from the required lead.

PROBLEMS

The following milling machine problems are based on Fig. 201. Determine the numbers of teeth in the gears A, B, C, and D in order to cut the following leads.

1. T = lead to be cut. Lead screw has 4 threads per inch.
2. M = lead to be cut. Lead screw has 4 threads per inch.
3. L = lead to be cut. Lead screw has 4 threads per inch.
4. N = lead to be cut. Lead screw has 5 threads per inch.
5. P = lead to be cut. Lead screw has 5 threads per inch.
6. Q = lead to be cut. Lead screw has 5 threads per inch.

VARIABLES

Prob.	Sym.	No 1	No 2	No 3	No 4	No. 5	No. 6
1	T	3.139	3 256	3 368	3 467	3.568	3.687
2	M	80 134	80 267	80 376	80 486	80 553	80.678
3	L	50.781	51 872	52 563	53 824	54 965	55.936
4	N	51 915	52 873	53 631	54 727	55 681	56.897
5	P	23 537	23 781	23 927	24 345	24 569	24.783
6	Q	34 186	34 397	34 576	34 789	34.819	34.907

CONTINUED FRACTIONS APPLIED TO THE CUTTING OF CAMS ON A VERTICAL MILLING MACHINE

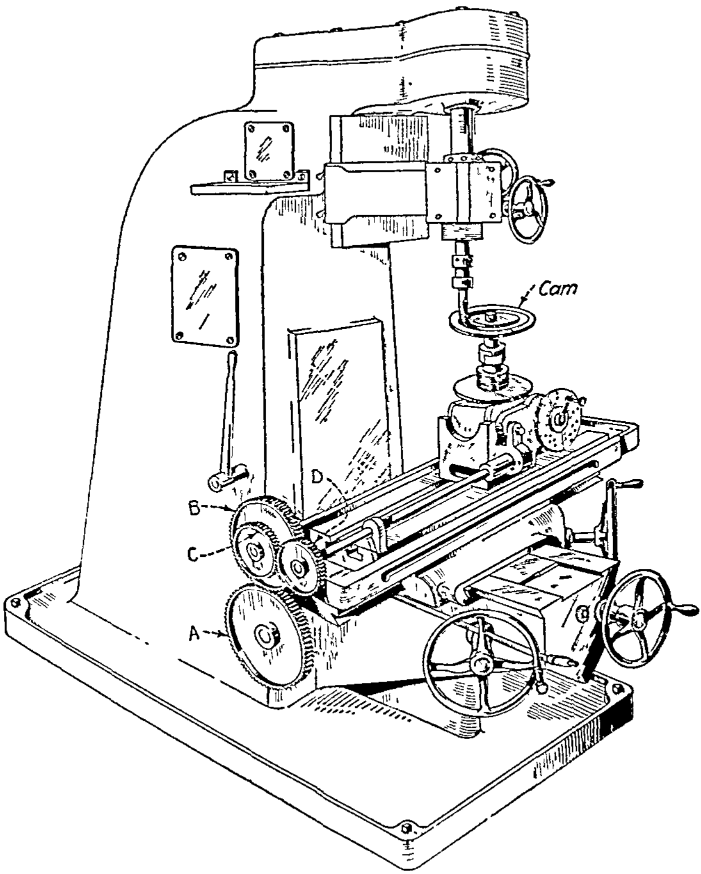


FIG. 202.

Figure 202 shows a vertical milling machine cutting a cam. The method of calculating the change gears for the vertical milling machine is the same as for the horizontal milling machine.

In order to cut cams similar to that shown in Fig. 203, three different sets of change gears are required, one set for each rise and one for each drop. When making the change from one set of gears to the next, care must be taken that the backlash is all in the same direction.

The lead of a cam, like that shown in Fig. 203, is the distance equal to the rise or drop in one complete revolution.

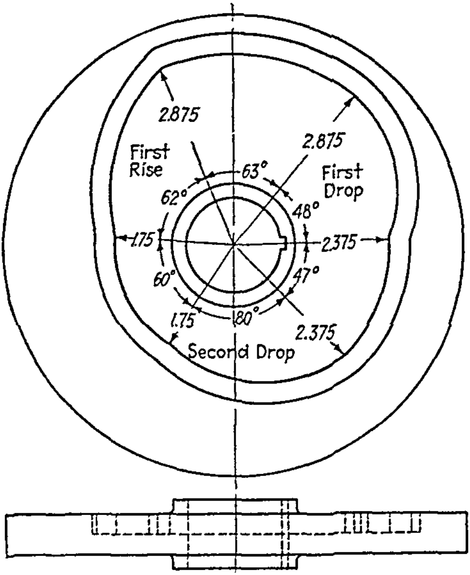


FIG. 203.

As shown in the figure, a $\begin{pmatrix}\text{rise}\\ \text{drop}\end{pmatrix}$ takes place from the terminal radius of a $\begin{pmatrix}\text{smaller}\\ \text{larger}\end{pmatrix}$ arc to the radius of the next $\begin{pmatrix}\text{larger}\\ \text{smaller}\end{pmatrix}$ arc. The rise or drop in a fractional part of a revolution is the difference in length of these two radii.

To determine the lead, when the rise or drop is continued for only a fractional part of a revolution, multiply the rise or drop by 360° and divide by the degrees included in the rise or drop.

The lead of the cam, thus obtained, is equal to the distance the table must move for one revolution of the spindle in the dividing head.

The change gears to be used on the vertical milling machines are calculated in the same manner as for a horizontal milling machine.

PROBLEMS

With reference to Figs. 202 and 203, N equals the number of threads per inch on the lead screw of the vertical mill.

1. Determine the number of teeth in the gears A, B, C, and D in order to mill the first rise.

2. Determine the number of teeth in the gears A, B, C, and D in order to mill the first drop.

3. Determine the number of teeth in the gears A, B, C, and D in order to mill the second drop.

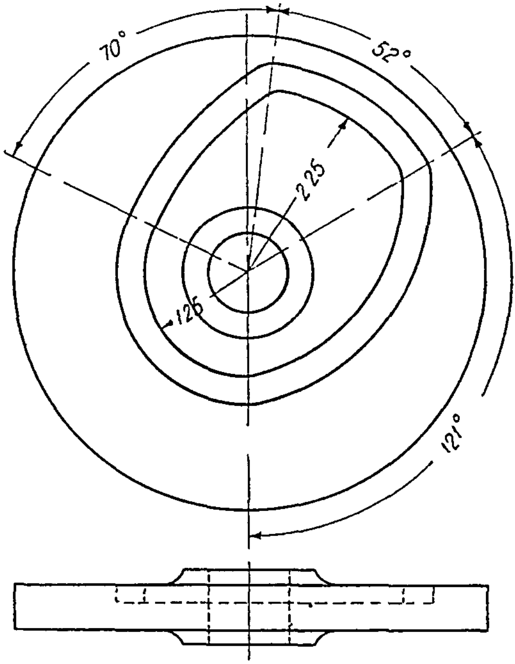


FIG. 204.

4. Referring to Fig. 204, determine the number of teeth in the gears A, B, C, and D in order to mill the rise.

5. Determine the number of teeth in the gears A, B, C, and D in order to mill the drop.

VARIABLES

Prob.	Sym.	No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1	N	3	4	5	6	7	8
2	N	3	4	5	6	7	8
3	N	3	4	5	6	7	8
4	N	10	9	8	7	6	5
5	N	10	9	8	7	6	5

CONTINUED FRACTIONS APPLIED TO CUTTING LEADS ON A HOBBING MACHINE

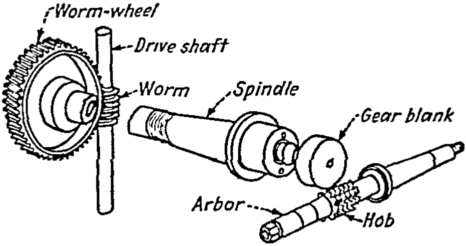


FIG. 205.

Spiral gears may be machined either on a milling machine or on a hobbing machine, but the latter is the better of the two because it generates the curvature of the tooth more perfectly. Figure 205 shows a gear blank mounted on a shaft, which is connected directly with the spindle, and the hob mounted on an arbor shaft in the cutting position. On the other end of the spindle is a worm and worm-wheel arrangement, the ratio of which is usually 15 to 1, 30 to 1, 60 to 1, or 90 to 1. This means that the driving shaft upon which the worm is mounted makes either 15, 30, 60, or 90 revolutions to 1 revolution of the spindle, depending upon the ratio of the worm and the worm wheel. The selection of the index gears which are located between the worm shaft and the arbor depends upon the number of teeth to be cut in the spiral gear and also upon the ratio of the worm and worm wheel of the machine. If the index change gears have equal numbers of teeth and the worm and worm-wheel *ratio is 30 to 1, the hob will make 30 revolutions* to 1 revolution of the spindle, thereby cutting 30 teeth in the gear. If the worm and worm-gear ratio is 60 to 1, and equal index gears are used, the hob will cut 60 teeth in the gear, etc. Suppose it were necessary to cut a 20-tooth gear. Since a 30 to 1 ratio of the worm and worm gear causes the hob to make 30 revolutions to 1 revolution of the spindle, thereby cutting 30 teeth, then inserting a 20-tooth gear on the worm shaft and a 30-tooth gear on the arbor shaft will cause the hob to make only 20 revolutions to 1 revolution of the spindle, thus cutting

only 20 teeth. Hence the number of teeth in the index change gears will be in the ratio of the number of teeth to be cut and the number of teeth in the worm wheel which is attached to the spindle.

In cutting the teeth in a spiral gear where a certain lead is required, all that is necessary is to compute the number of revolutions that the spindle makes and the number of revolutions that the hob makes.

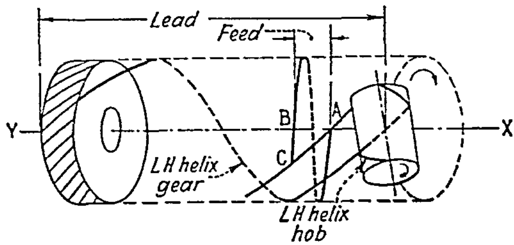


FIG. 206.

Figure 206 shows the relation between the helix of the spiral gear and the helix produced by advancing the hob a distance equal to the feed when the spiral gear makes one revolution. It also shows that the helix of the spiral gear and the helix of the hob are of like hands. The distance AB that the hob advances along the axis XY of the spiral gear when the spindle makes one revolution is called the feed. Since the points A and C are in the same groove of the helix of the spiral gear, a single-threaded hob makes as many revolutions in going from A to C on the helix ABC as there are number of teeth in the spiral gear. Since the hob goes only from A to B on the helix ABC for each revolution of the spiral gear, the hob makes a fractional part of a revolution less than the number of teeth in the spiral gear. Let the portion BC of the helix ABC represent the fractional part of a revolution that the hob lags to each revolution of the spiral gear. Since the lead of the helix of the spiral gear is equal to the distance that the helix advances in one revolution, then the sum of the portions BC of the helix ABC when the hob moves from right to left a distance equal to the lead will completely encircle the spiral gear. From this it is evident that the number of revolutions

that the hob lags when it is moved a distance equal to the lead is equal to the number of teeth in the spiral gear. Then the total number of revolutions that the hob makes will be equal to the lead divided by the feed multiplied by the number of teeth in the gear, minus a number of revolutions equal to the number of teeth in the gear.

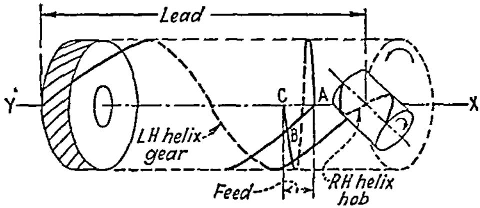


FIG. 207.

It is also evident in Fig. 207, which shows a spiral gear and hob of opposite hands, that the hob in tracing the helix of the feed from A to C makes more revolutions by a fractional amount than there are teeth in the gear. In this case, by the same reasoning applied to Fig. 206, it may be shown that the number of revolutions of the hob is equal to the lead divided by the feed, multiplied by the number of teeth in the gear, plus the number of teeth in the gear. From the foregoing discussion, the following formula is derived for finding the number of teeth in the index gears:

Let L = lead, F = feed, N = number of teeth in spiral gear

R = ratio of worm wheel to worm

$$\text{Revolutions of spindle} = \frac{L}{F}$$

$$\text{Revolutions of worm shaft} = \frac{RL}{F}$$

$$\text{Revolutions of hob} = \frac{LN}{F} \pm N$$

where the plus sign is used for opposite hands and the minus sign is used for like hands.

$$\frac{\text{Number of teeth in gear on hob}}{\text{Number of teeth in gear on worm shaft}} =$$

$$\frac{\text{revolutions of worm shaft}}{\text{revolutions of hob}} = \frac{R\frac{L}{F}}{\frac{L}{F}N \pm N} = \frac{\frac{RL}{F}}{\frac{LN \pm FN}{F}} =$$

$$\frac{RL}{N(L \pm F)} = \frac{\text{gear on arbor}}{\text{gear on worm shaft}} = \frac{\text{driver gears}}{\text{driven gears}}.$$

To determine the proper numbers of teeth for the feed gears, first determine the distance that the carriage supporting the hob moves when the feed gears have equal numbers of teeth. This distance is .0375 in. on some machines and .075 in. and .125 in. on others. For a carriage moving .075 with equal numbers of teeth in the feed gears, a feed of .001 will be obtained by using gears having a teeth ratio of 1 to 75 (driver to driven); a feed of .020 will be obtained by using gears having a teeth ratio of 20 to 75; etc. Thus the ratio for the feed gears is

$$\frac{\text{Driver gears}}{\text{Driven gears}} = \frac{F}{.075}.$$

For a hobbing machine with the carriage moving .125 in. for equal teeth in the feed gears, the ratio is:

$$\frac{\text{Driver gears}}{\text{Driven gears}} = \frac{F}{.125}.$$

The application of continued fractions for obtaining the proper numbers of teeth in the index and feed gears can best be understood by solving a typical problem.

Example: Let it be required to cut a lead on a 5-diametral pitch spiral gear having a helix angle of 45° and 12 teeth by a hobbing machine having a 30 to 1 ratio and a carriage movement of .075 in. for equal numbers of teeth in the feed change gears. The helix of the hob and the spiral gear are of opposite hands. The feed of the carriage is .020 in.

Solution: The lead must first be computed by using the spiral-gear formula $L = \pi D \cot \alpha_g$ where the pitch diameter

$$D = \frac{N \sec \alpha_g}{P_n}.$$

In this problem, $D = \dfrac{12 \sec 45°}{5} = 3.39410$ and

$$L = 3.1416 \times 3.39410 \times 1 = 10.66290.$$

Since the helix of the gear and the helix of the hob are of opposite hands, the plus sign must be used in the index change gear formula,

$$\frac{\text{Teeth of index driver gears}}{\text{Teeth of index driven gears}} = \frac{RL}{N(L+F)} =$$
$$\frac{30 \times 10.6629}{12(10.6629 + .020)} = \frac{53.3145}{21.3658} = \frac{533145}{213658}.$$

This fraction expressed as continued fractions is as follows:

		2	52	1	10	1	2	3	2	7
$\frac{1}{0}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{626}{105}$	$\frac{267}{107}$	$\frac{2932}{1175}$	$\frac{3199}{1282}$	$\frac{9330}{3739}$	$\frac{31189}{12499}$	$\frac{71708}{28737}$	$\frac{533145}{213658}$

Carrying out the usual work of combining convergents, the first factorable number is found to be

$$\frac{2932 - 267}{1175 - 107} = \frac{2665}{1068} = \frac{12 \times 89}{41 \times 65} = \frac{24 \times 89}{82 \times 65} = \frac{\text{driver}}{\text{driven}}.$$

Solving for the lead in the formula $\dfrac{\text{Driver}}{\text{Driven}} = \dfrac{X}{Y} = \dfrac{RL}{N(L \pm F)}$ gives for opposite hands $L = \dfrac{NFX}{RY - NX}$ and for like hands $L = \dfrac{NFX}{NX - RY}$.

Applying this formula to check the lead in the foregoing problem:

$$L = \frac{NFX}{RY - NX} = \frac{12 \times .02 \times 2665}{30 \times 1068 - 12 \times 2665} = 10.6600$$

which gives a variation from the original lead of

$$10.6629 - 10.6600 = .0029.$$

For spiral gears having angular shafts this error would be permissible, but for spiral gears having parallel shafts this error is too great and necessitates solving for a new set of feed gears with the aid of continued fractions which is carried out as follows:

Solving for the feed in the equation $\frac{X}{Y} = \frac{RL}{NL \pm NF}$ gives for opposite hands $F = \frac{RLY}{NX} - L$ and for like hands

$$F = L - \frac{RLY}{NX}.$$

Solving for the new feed produced by using the index gears just computed gives:

$$F = \frac{RLY}{NX} - L = \frac{30 \times 10.6629 \times 1068}{2665 \times 12} - 10.6629 = .0200055.$$

For feed gears,

$$\frac{\text{Driver gears}}{\text{Driven gears}} = \frac{F}{.075} = \frac{.0200055}{.075} = \frac{200055}{750000}.$$

The quotients are obtained in the usual manner as follows:

200055	750000	3
149835	600165	1
50220	149835	2
49395	100440	1
825	49395	59
720	48675	1
105	720	6
90	630	1
15	90	6
	90	
	0	

The foregoing work not only gives the quotient but shows that 15 is the highest common factor. Since the last convergent always gives the original ratio in its reduced form, the

numerator and denominator of the last convergent in this case must be multiplied by 15 to give the original numbers.

		3	1	2	1	59	1	6	1	6
$\frac{1}{0}$	$\frac{0}{1}$.	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{3}{11}$	$\frac{4}{15}$	$\frac{259}{896}$	$\frac{243}{911}$	$\frac{1697}{6362}$	$\frac{1940}{7273}$	$\frac{13337}{50000}$

$$\textit{Check:}\ \frac{13{,}337 \times 15}{50{,}000 \times 15} = \frac{200{,}055}{750{,}000}.$$

Starting with $\frac{1940}{7273}$ and combining the convergents in the usual manner to obtain a series of lesser values gives the fraction $\frac{1705}{6392}$ in the sixth step, which is factorable.

$$\frac{1705}{6392} = \frac{31 \times 55}{68 \times 94} = \frac{U}{M} = \frac{\text{driver}}{\text{driven}}$$

where $\frac{U}{M}$ is the new ratio of the feed gears (drivers to driven).

Solving for the feed obtained with this new ratio:

$$\frac{F}{.075} = \frac{U}{M} = \frac{\text{driver}}{\text{driven}}$$

or

$$F = \frac{.075 \times U}{M} = \frac{.075 \times 1705}{6392} = .02000547$$

Substituting this value of the feed in the lead formula gives

$$L = \frac{NFX}{RY - NX} = \frac{12 \times .02000547 \times 2665}{30 \times 1068 - 12 \times 2665} = 10.662915$$

which is a variation from the original lead of only .000015.

The variation in the helix angle due to this small variation in the lead may be computed from the formula

$$\csc\ \alpha_g = \frac{LP_n}{\pi N}$$

which is derived from the spiral-gear formulas $D = \frac{N \sec \alpha_g}{P_n}$ and $L = \pi D \cot \alpha_g$.

In this case,

$$\csc \alpha_g = \frac{LP_n}{\pi N} = \frac{10.66292 \times 5}{3.1416 \times 12} = 1.4142$$

which, using a five-place table, gives $\alpha_g = 45°$, which means that there is no appreciable variation in the helix angle.

PROBLEMS

Determine the index change gears (and the feed change gears, only when the shafts of the spiral gears are parallel) required on a hobbing machine (as in Fig. 205) according to the data given below. Consider that the worm and worm-wheel ratio is 30 to 1 and that, when equal numbers of teeth are used for the feed gears, the carriage supporting the hob moves forward .075 in. for 1 revolution of the spindle.

1. Spiral gear to have 37 teeth, helix angle of 35° 40′, and a diametral pitch of 8, assuming that the shafts of the spiral gears are angular. The cutting feed is to be .034 in. per revolution of the spindle.
2. Spiral gear to have 22 teeth, helix angle of 15° 45′, and a diametral pitch of 4, assuming that the shafts of the spiral gears are parallel. The cutting feed is to be .054 in. per revolution of the spindle.
3. Spiral gear to have 50 teeth, helix angle of 48° 20′, and a diametral pitch of 12, assuming that the shafts of the spiral gears are angular. The feed is to be .045 in. per revolution of the spindle.
4. Spiral gear to have 42 teeth, helix angle of 18° 35′, and a diametral pitch of 7, assuming that the shafts of the spiral gears are parallel. The feed is to be .025 in. per revolution of the spindle.
5. Spiral gear to have 25 teeth, helix angle of 36° 25′, and a diametral pitch of 6, assuming that the shafts of the spiral gears are angular. The feed is to be .048 in. per revolution of the spindle.
6. Spiral gear to have 30 teeth, helix angle 23° 40′, and a diametral pitch of 5, assuming the shafts to be parallel. The feed is to be .062 in. per revolution of the spindle.

FACTOR TABLE 20 to 299

Number	Factors	Number	Factors	Number	Factors	Number	Factors
20	4–5	91	7–13	159	3–53	226	2–113
21	3–7	92	4–23	160	10–16	228	12–19
22	2–11	93	3–31	161	7–23	230	10–23
24	4–6	94	2–47	162	9–18	231	11–21
25	5–5	95	5–19	164	4–41	232	8–29
26	2–13	96	8–12	165	11–15	234	13–18
27	3–9	98	7–14	166	2–83	235	5–47
28	4–7	99	9–11	168	8–21	236	4–59
30	5–6	100	10–10	169	13–13	237	3–79
32	4–8	102	6–17	170	10–17	238	14–17
33	3–11	104	8–13	171	9–19	240	15–16
34	2–17	105	7–15	172	4–43	242	11–22
35	5–7	106	2–53	174	6–29	243	9–27
36	4–9	108	9–12	175	7–25	244	4–61
38	2–19	110	10–11	176	11–16	245	7–35
39	3–13	111	3–37	177	3–59	246	6–41
40	5–8	112	8–14	178	2–89	247	13–19
42	6–7	114	6–19	180	12–15	248	8–31
44	4–11	115	5–23	182	13–14	249	3–83
45	5–9	116	4–29	183	3–61	250	10–25
46	2–23	117	9–13	184	8–23	252	14–18
48	6–8	118	2–59	185	5–37	253	11–23
49	7–7	119	7–17	186	6–31	255	15–17
50	5–10	120	10–12	187	11–17	256	16–16
51	3–17	121	11–11	188	4–47	258	6–43
52	4–13	122	2–61	189	9–21	259	7–37
54	6–9	123	3–41	190	10–19	260	13–20
55	5–11	124	4–31	192	12–16	261	9–29
56	7–8	125	5–25	194	2–97	264	11–24
57	3–19	126	9–14	195	13–15	265	5–53
58	2–29	128	8–16	196	14–14	266	14–19
60	6–10	129	3–43	198	11–18	267	3–89
62	2–31	130	10–13	200	10–20	268	4–67
63	7–9	132	11–12	201	3–67	270	15–18
64	8–8	133	7–19	202	2–101	272	16–17
65	5–13	134	2–67	203	7–29	273	13–21
66	6–11	135	5–27	204	12–17	275	11–25
68	4–17	136	8–17	205	5–41	276	12–23
69	3–23	138	6–23	207	9–23	279	9–31
70	7–10	140	10–14	208	13–16	280	14–20
72	8–9	141	3–47	209	11–19	282	6–47
74	2–37	142	2–71	210	14–15	284	4–71
75	5–15	143	11–13	212	4–53	285	15–19
76	4–19	144	12–12	213	3–71	286	13–22
77	7–11	145	5–29	214	2–107	287	7–41
78	6–13	146	2–73	215	5–43	288	16–18
80	8–10	147	3–49	216	12–18	289	17–17
81	9–9	148	4–37	217	7–31	290	10–29
82	2–41	150	10–15	218	2–109	291	3–97
84	7–12	152	8–19	219	3–73	292	4–73
85	5–17	153	9–17	220	11–20	294	6–49
86	2–43	154	11–14	221	13–17	295	5–59
87	3–29	155	5–31	222	6–37	296	8–37
88	8–11	156	12–13	224	14–16	297	11–27
90	9–10	158	2–79	225	15–15	299	13–23

FACTOR TABLE.—(*Continued*) 300 to 627

Number	Factors	Number	Factors	Number	Factors	Number	Factors
300	15– 20	374	17– 22	459	17– 27	540	20– 27
301	7– 43	375	15– 25	460	20– 23	544	17– 32
303	3–101	376	8– 47	462	21– 22	545	5–109
304	16– 19	377	13– 29	464	16– 29	546	21– 26
305	5– 61	378	18– 21	465	15– 31	549	9– 61
306	17– 18	380	19– 20	468	18– 26	550	22– 25
308	14– 22	384	16– 24	469	7– 67	551	19– 29
309	3–103	385	11– 35	470	10– 47	552	23– 24
310	10– 31	387	9– 43	472	8– 59	553	7– 79
312	13– 24	388	4– 97	473	11– 43	555	15– 37
315	15– 21	390	15– 26	474	6– 79	558	18– 31
316	4– 79	391	17– 23	475	19– 25	559	13– 43
318	6– 53	392	14– 28	476	17– 28	560	20– 28
319	11– 29	395	5– 79	477	9– 53	561	17– 33
320	16– 20	396	18– 22	480	20– 24	564	12– 47
321	3–107	399	19– 21	481	13– 37	565	5–113
322	14– 23	400	20– 20	483	21– 23	567	21– 27
323	17– 19	402	6– 67	484	22– 22	568	8– 71
324	18– 18	403	13– 31	485	5– 97	570	19– 30
325	13– 25	404	4–101	486	18– 27	572	22– 26
327	3–109	405	15– 27	488	8– 61	574	14– 41
328	8– 41	406	14– 29	490	10– 49	575	23– 25
329	7– 47	407	11– 37	492	12– 41	576	24– 24
330	15– 22	408	17– 24	493	17– 29	578	17– 34
332	4– 83	410	10– 41	494	19– 26	580	20– 29
333	9– 37	413	7– 59	495	15– 33	581	7– 83
335	5– 67	414	18– 23	496	16– 31	582	6– 97
336	16– 21	415	5– 83	497	7– 71	583	11– 53
338	13– 26	416	16– 26	498	6– 83	584	8– 73
339	3–113	418	19– 22	500	20– 25	585	15– 39
340	17– 20	420	20– 21	504	21– 24	588	21– 28
341	11– 31	423	9– 47	505	5–101	589	19– 31
342	18– 19	425	17– 25	506	22– 23	590	10– 59
343	7– 49	426	6– 71	507	13– 39	592	16– 37
344	8– 43	427	7– 61	510	17– 30	594	22– 27
345	15– 23	428	4–107	511	7– 73	595	17– 35
348	12– 29	429	13– 33	512	16– 32	598	23– 26
350	14– 25	430	10– 43	513	19– 27	600	24– 25
351	13– 27	432	18– 24	516	12– 43	602	14– 43
352	16– 22	434	14– 31	517	11– 47	603	9– 67
354	6– 59	435	15– 29	518	14– 37	605	11– 55
355	5– 71	436	4–109	520	13– 40	606	6–101
356	4– 89	437	19– 23	522	18– 29	608	19– 32
357	17– 21	438	6– 73	525	21– 25	609	21– 29
360	18– 20	440	20– 22	527	17– 31	610	10– 61
361	19– 19	441	9– 49	528	22– 24	611	13– 47
363	11– 33	442	17– 26	529	23– 23	612	17– 36
364	14– 26	444	12– 37	530	10– 53	615	15– 41
365	5– 73	445	5– 89	531	9– 59	616	22– 28
366	6– 61	448	16– 28	532	19– 28	620	20– 31
368	16– 23	450	18– 25	533	13– 41	621	23– 27
369	9– 41	451	11– 41	534	6– 89	623	7– 89
370	10– 37	452	4–113	535	5–107	624	24– 26
371	7– 53	455	13– 35	536	8– 67	625	25– 25
372	12– 31	456	19– 24	539	11– 49	627	19– 33

FACTOR TABLE.—(*Continued*) 629 to 1000

Number	Factors	Number	Factors	Number	Factors	Number	Factors
629	17– 37	715	13– 55	808	8–101	902	22– 41
630	21– 30	720	24– 30	810	27– 30	903	21– 43
632	8– 79	721	7–103	812	28– 29	904	8–113
636	12– 53	722	19– 38	814	22– 37	909	9–101
637	13– 49	725	25– 29	816	24– 34	910	26– 35
638	22– 29	726	22– 33	817	19– 43	912	24– 38
639	9– 71	728	26– 28	819	21– 39	913	11– 83
640	20– 32	729	27– 27	820	20– 41	915	15– 61
642	6–107	730	10– 73	825	25– 33	918	27– 34
644	23– 28	731	17– 43	826	14– 59	920	23– 40
645	15– 43	732	12– 61	828	23– 36	923	13– 71
646	19– 34	735	21– 35	830	10– 83	924	28– 33
648	24– 27	736	23– 32	832	26– 32	925	25– 37
649	11– 59	737	11– 67	833	17– 49	928	29– 32
650	25– 26	738	18– 41	836	22– 38	930	30– 31
651	21– 31	740	20– 37	837	27– 31	931	19– 49
654	6–109	741	19– 39	840	28– 30	935	17– 55
656	16– 41	742	14– 53	841	29– 29	936	24– 39
657	9– 73	744	24– 31	845	13– 65	938	14– 67
658	14– 47	747	9– 83	846	18– 47	940	20– 47
660	12– 55	748	44– 17	847	11– 77	943	23– 41
663	17– 39	749	7–107	848	16– 53	944	16– 59
664	8– 83	750	25– 30	850	25– 34	945	27– 35
665	19– 35	752	16– 47	851	23– 37	946	22– 43
666	18– 37	754	26– 29	852	12– 71	948	12– 79
667	23– 29	756	27– 28	854	14– 61	949	13– 73
670	10– 67	759	23– 33	855	19– 45	950	25– 38
671	11– 61	760	19– 40	856	8–107	952	28– 34
672	24– 28	763	7–109	858	22– 39	954	18– 53
675	25– 27	765	17– 45	860	20– 43	957	29– 33
676	26– 26	767	13– 59	861	21– 41	960	30– 32
678	6–113	768	24– 32	864	27– 32	961	31– 31
679	7– 97	770	22– 35	867	17– 51	962	26– 37
680	17– 40	774	18– 43	868	28– 31	963	9–107
682	22– 31	775	25– 31	869	11– 79	966	23– 42
684	19– 36	776	8– 97	870	29– 30	968	22– 44
686	14– 49	777	21– 37	871	13– 67	969	19– 51
688	16– 43	779	19– 41	872	8–109	970	10– 97
689	13– 53	780	20– 39	873	9– 97	972	27– 36
690	23– 30	781	11– 71	874	23– 38	975	25– 39
693	21– 33	782	23– 34	875	25– 35	976	16– 61
696	24– 29	783	27– 29	876	12– 73	979	11– 89
697	17– 41	784	28– 28	880	22– 40	980	28– 35
700	25– 28	790	10– 79	882	21– 42	981	9–109
702	26– 27	792	24– 33	884	26– 34	984	24– 41
703	19– 37	793	13– 61	885	15– 59	986	29– 34
704	22– 32	795	15– 53	888	24– 37	987	21– 47
705	15– 47	798	21– 38	890	10– 89	988	26– 38
707	7–101	799	17– 47	891	27– 33	989	23– 43
708	12– 59	800	25– 32	893	19– 47	990	30– 33
710	10– 71	801	9– 89	896	28– 32	992	31– 32
711	9– 79	803	11– 73	897	23– 39	994	14– 71
712	8– 89	804	12– 67	899	29– 31	996	12– 83
713	23– 31	805	23– 35	900	30– 30	999	27– 37
714	21– 34	806	26– 31	901	17– 53	1000	25– 40

FACTOR TABLE.—(*Continued*) 1001 to 1411

Number	Factors	Number	Factors	Number	Factors	Number	Factors
1001	13–77	1102	29–38	1204	28–43	1308	12–109
03	17–59	04	23–48	06	18–67	09	17–77
05	15–67	05	17–65	07	17–71	11	23–57
07	19–53	06	14–79	09	31–39	12	32–41
08	28–36	07	27–41	10	22–55	13	13–101
10	10–101	10	30–37	12	12–101	14	18–73
12	22–46	11	11–101	15	27–45	16	28–47
14	26–39	13	21–53	16	32–38	20	33–40
15	29–35	16	31–36	18	29–42	23	27–49
20	20–51	18	26–43	19	23–53	25	25–53
22	14–73	20	32–35	20	20–61	26	26–51
23	31–33	21	19–59	21	33–37	28	16–83
24	32–32	22	33–34	22	26–47	30	35–38
25	25–41	25	25–45	24	24–51	32	36–37
26	27–38	27	23–49	25	35–35	33	31–43
27	13–79	28	24–47	30	30–41	34	29–46
29	21–49	30	10–113	32	28–44	35	15–89
32	24–43	31	29–39	35	19–65	39	13–103
34	22–47	33	11–103	39	21–59	40	20–67
35	23–45	34	27–42	40	40–31	42	22–61
36	28–37	36	16–71	41	17–73	43	17–79
37	17–61	39	17–67	42	27–46	44	32–42
40	26–40	40	20–57	43	11–113	49	19–71
44	29–36	44	26–44	45	15–83	50	30–45
45	19–55	47	31–37	46	14–89	52	26–52
50	30–35	48	28–41	47	29–43	53	33–41
53	27–39	50	25–46	48	32–39	56	12–113
54	31–34	52	32–36	50	25–50	57	23–59
56	32–33	55	33–35	54	33–38	58	14–97
58	23–46	56	34–34	58	34–37	60	34–40
60	20–53	57	13–89	60	30–42	63	29–47
62	18–59	59	19–61	61	13–97	64	31–44
64	28–38	60	20–58	64	16–79	65	35–39
65	15–71	61	27–43	65	23–55	68	36–38
66	26–41	62	14–83	69	27–47	69	37–37
67	11–97	64	12–97	71	31–41	72	28–49
68	12–89	66	22–53	72	24–53	75	25–55
70	10–107	68	16–73	73	19–67	76	32–43
71	21–51	70	30–39	74	26–49	77	27–51
72	16–67	73	23–51	75	25–51	78	26–53
73	29–37	75	25–47	76	29–44	80	23–60
75	25–43	76	28–42	78	18–71	86	33–42
78	22–49	77	11–107	80	32–40	87	19–73
79	13–83	78	31–38	81	21–61	91	13–107
80	30–36	80	20–59	84	12–107	92	29–48
81	23–47	83	13–91	87	33–39	94	34–41
83	19–57	84	32–37	88	28–46	95	31–45
85	31–35	85	15–79	90	30–43	1400	35–40
88	32–34	88	33–36	92	34–38	03	23–61
89	33–33	89	29–41	95	35–37	04	36–39
90	10–109	90	34–35	96	36–36	06	37–38
92	28–39	96	26–46	98	22–59	07	21–67
95	15–73	97	21–57	1300	26–50	08	32–44
98	18–61	99	11–109	02	31–42	10	30–47
1100	20–55	1200	30–40	05	29–45	11	17–83

Factor Table.—(*Continued*) 1414 to 1849

Number	Factors	Number	Factors	Number	Factors	Number	Factors
1414	14–101	1518	33–46	1632	34–48	1740	30–58
16	24–59	19	31–49	33	23–71	42	26–67
17	13–109	20	38–40	34	38–43	43	21–83
19	33–43	21	39–39	35	15–109	44	16–109
20	20–71	25	25–61	38	39–42	46	18–97
21	29–49	26	14–109	40	40–41	48	38–46
22	18–79	30	30–51	43	31–53	49	33–53
24	16–89	33	21–73	45	35–47	50	35–50
25	25–57	34	26–59	47	27–61	51	17–103
26	31–46	36	32–48	49	17–97	52	24–73
28	34–42	37	29–53	50	33–50	55	39–45
30	26–55	39	27–57	52	28–59	60	40–44
31	27–53	40	35–44	53	29–57	63	41–43
35	35–41	41	23–67	56	36–46	64	42–42
40	36–40	47	17–91	59	21–79	67	31–57
43	37–39	48	36–43	60	20–83	68	34–52
44	38–38	50	31–50	64	32–52	69	29–61
45	17–85	51	33–47	65	37–45	70	30–59
49	23–63	52	16–97	66	34–49	71	23–77
50	29–50	54	37–42	72	38–44	75	25–71
52	33–44	58	38–41	74	31–54	76	37–48
55	15–97	60	39–40	75	25–67	80	20–89
56	28–52	62	22–71	77	39–43	82	33–54
57	31–47	64	34–46	79	23–73	85	35–51
58	27–54	66	29–54	80	40–42	86	38–47
60	20–73	68	32–49	81	41–41	92	32–56
62	34–43	75	35–45	82	29–58	94	39–46
63	19–77	77	19–83	83	33–51	98	31–58
64	24–61	80	20–79	90	26–65	1800	40–45
69	13–113	81	31–51	91	19–89	02	34–53
70	35–42	82	14–113	92	36–47	04	41–44
72	32–46	84	36–44	94	22–77	05	19–95
74	22–67	86	26–61	95	15–113	06	42–43
75	25–59	87	23–69	96	32–53	08	16–113
76	36–41	90	30–53	1700	34–50	09	27–67
79	29–51	91	37–43	01	27–63	13	37–49
80	37–40	93	27–59	02	37–46	15	33–55
82	38–39	95	29–55	04	24–71	17	23–79
84	28–53	96	38–42	05	31–55	18	18–101
85	27–55	98	34–47	08	28–61	19	17–107
88	31–48	99	39–41	10	38–45	20	35–52
91	21–71	1600	40–40	11	29–59	24	32–57
94	18–83	02	18–89	12	16–107	25	25–73
95	23–65	05	15–107	15	35–49	26	22–83
96	34–44	06	22–73	16	39–44	27	29–63
98	14–107	08	24–67	17	17–101	29	31–59
1500	30–50	10	35–46	20	40–43	30	30–61
01	19–79	12	31–52	22	41–42	33	39–47
04	32–47	15	19–85	25	25–69	36	36–51
05	35–43	16	16–101	28	32–54	40	40–46
08	29–52	17	33–49	29	19–91	43	19–97
12	36–42	20	36–45	34	34–51	45	41–45
13	17–89	24	29–56	36	31–56	46	26–71
15	15–101	25	25–65	38	22–79	48	42–44
17	37–41	28	37–44	39	37–47	49	43–43

FACTOR TABLE.—(*Continued*) 1850 to 2324

Number	Factors	Number	Factors	Number	Factors	Number	Factors
1850	37–50	1971	27–73	2088	36–58	2205	45–49
55	35–53	72	34–58	90	38–55	08	46–48
56	32–58	74	42–47	91	41–51	09	47–47
60	31–60	75	25–79	93	23–91	10	34–65
62	38–49	76	38–52	2100	42–50	11	33–67
63	27–69	78	43–46	06	39–54	12	28–79
69	21–89	80	44–45	07	43–49	14	41–54
70	34–55	84	32–62	08	34–62	20	37–60
72	39–48	88	28–71	09	37–57	22	22–101
75	25–75	89	39–51	12	44–48	23	39–57
76	28–67	92	24–83	15	45–47	25	25–89
80	40–47	95	35–57	16	46–46	26	42–53
81	33–57	98	37–54	17	29–73	31	23–97
85	29–65	2000	40–50	20	40–53	32	31–72
86	41–46	01	29–69	21	21–101	33	29–77
87	37–51	02	26–77	24	36–59	36	43–52
88	32–59	06	34–59	25	25–85	40	40–56
90	35–54	09	41–49	28	38–56	41	27–83
91	31–61	10	30–67	30	30–71	42	38–59
92	43–44	13	33–61	32	41–52	44	44–51
96	24–79	14	38–53	33	27–79	47	21–107
98	26–73	15	31–65	34	22–97	50	45–50
1900	38–50	16	42–48	35	35–61	54	46–49
04	34–56	20	20–101	36	24–89	55	41–55
08	36–53	21	43–47	39	31–69	56	47–48
09	23–83	24	44–46	40	20–107	57	37–61
11	39–49	25	45–45	42	42–51	60	20–113
14	33–58	28	39–52	44	32–67	61	19–119
17	27–71	30	35–58	45	39–55	62	39–58
19	19–101	33	19–107	46	37–58	63	31–73
20	32–60	34	18–113	47	19–113	68	42–54
21	17–113	35	37–55	50	43–50	72	32–71
22	31–62	37	21–97	56	44–49	75	35–65
24	37–52	40	40–51	58	26–83	77	33–69
25	35–55	44	28–73			78	34–67
26	18–107	46	33–62	60	45–48	79	43–53
27	41–47	47	23–89	62	46–47	80	40–57
32	42–46	48	32–64	63	21–103	88	44–52
35	43–45	50	41–50	66	38–57	89	21–109
36	44–44	52	38–54	70	35–62	91	29–79
38	38–51	54	26–79	73	41–53	94	37–62
40	20–97	58	42–49	75	25–87	95	45–51
43	29–67	59	29–71	76	34–64	96	41–56
44	36–54	64	43–48	78	33–66	2300	46–50
47	33–59	65	35–59	80	20–109	01	39–59
50	39–50	67	39–53	83	37–59	03	49–47
52	32–61	68	44–47	84	42–52	04	48–48
53	31–63	70	45–46	85	23–95	10	42–55
55	23–85	71	19–109	87	27–81	12	34–68
57	19–103	72	37–56	90	30–73	14	26–89
58	22–89	74	34–61	93	43–51	18	38–61
60	40–49	75	25–83	96	36–61	20	29–80
61	37–53	77	31–67	2200	44–50	22	43–54
62	18–109	79	33–63	01	31–71	23	23–101
68	41–48	80	40–52	04	38–58	24	28–83

FACTOR TABLE.—(*Continued*) 2325 to 2825

Number	Factors	Number	Factors	Number	Factors	Number	Factors
2325	31– 75	2444	47– 52	2574	39– 66	2700	50– 54
28	24– 97	48	48– 51	75	25–103	01	37– 73
31	37– 63	49	31– 79	76	46– 56	03	51– 53
32	44– 53	50	49– 50	80	43– 60	04	52– 52
36	32– 73	51	43– 57	81	29– 89	06	33– 82
37	41– 57	57	39– 63	83	41– 63	09	43– 63
40	45– 52	60	41– 60	84	38– 68	12	24–113
43	33– 71	61	23–107	85	47– 55	14	46– 59
45	35– 67	64	44– 56	90	37– 70	16	28– 97
46	46– 51	65	29– 85	92	48– 54	20	40– 68
49	29– 81	70	38– 65	96	44– 59	25	25–109
50	47– 50	75	45– 55	97	49– 53	26	47– 58
52	48– 49	78	42– 59	99	23–113	27	27–101
54	22–107	79	37– 67	2600	50– 52	28	44– 62
56	38– 62	80	40– 62	01	51– 51	30	42– 65
60	40– 59	82	34– 73	04	42– 62	36	48– 57
65	43– 55	84	36– 69	07	33– 79	37	23–119
66	26– 91	85	35– 71	10	45– 58	38	37– 74
68	37– 64	86	22–113	13	39– 67	39	33– 83
69	23–103	90	30– 83	16	24–109	44	49– 56
70	30– 79	91	47– 53	18	34– 77	45	45– 61
73	21–113	92	28– 89	19	27– 97	47	41– 67
75	25– 95	94	43– 58	22	46– 57	50	50– 55
76	44– 54	96	48– 52	23	43– 61	52	43– 64
78	41– 58	99	49– 51	24	41– 64	54	51– 54
79	39– 61	2500	50– 50	25	35– 75	55	29– 95
80	34– 70	01	41– 61	26	26–101	56	52– 53
85	45– 53	07	23–109	27	37– 71	59	31– 89
87	31– 77	08	44– 57	28	36– 73	60	46– 60
92	46– 52	11	31– 81	32	47– 56	65	35– 79
94	38– 63	16	34– 74	35	31– 85	69	39– 71
97	47– 51	20	45– 56	39	29– 91	72	44– 63
98	22–109	22	26– 97	40	48– 55	73	47– 59
2400	48– 50	23	29– 87	46	49– 54	74	38– 73
01	49– 49	25	25–101	50	50– 53	75	37– 75
03	27– 89	28	32– 79	52	39– 68	81	27–103
05	37– 65	30	46– 55	55	45– 59	82	26–107
07	29– 83	35	39– 65	56	32– 83	84	48– 58
08	43– 56	37	43– 59	60	38– 70	88	41– 68
09	33– 73	38	47– 54	64	37– 72	90	45– 62
12	36– 67	41	33– 77	65	41– 65	93	49– 57
14	34– 71	42	41– 62	66	43– 62	95	43– 65
15	35– 69	44	48– 53	68	46– 58	2800	50– 56
	Out	46	38– 67	70	30– 89	05	51– 55
18	39– 62	48	49– 52	73	33– 81	06	46– 61
19	41– 59	50	50– 51	75	25–107	08	52– 54
20	44– 55	52	44– 58	79	47– 57	09	53– 53
24	24–101	53	37– 69	80	40– 67	12	38– 74
25	25– 97	55	35– 73	84	44– 61	13	29– 97
30	45– 54	56	36– 71	86	34– 79	14	42– 67
32	38– 64	60	40– 64	88	48– 56	16	44– 64
36	42– 58	62	42– 61	91	39– 69	20	47– 60
38	46– 53	65	45– 57	95	49– 55	21	31– 91
40	40– 61	68	24–107	97	31– 87	22	34– 83
42	33– 74	73	31– 83	98	38– 71	25	25–113

FACTOR TABLE.—(*Continued*) 2828 to 3363

Number	Factors	Number	Factors	Number	Factors	Number	Factors
2828	28–101	2958	51– 58	3087	49– 63	3230	38– 85
29	41– 69	60	40– 74	94	34– 91	32	32–101
32	48– 59	61	47– 63	96	43– 72	33	53– 61
34	26–109	64	52– 57	3100	50– 62	34	49– 66
35	45– 63	67	43– 69	02	47– 66	37	39– 83
38	43– 66	68	53– 56	03	29–107	39	41– 79
40	40– 71	70	54– 55	04	32– 97	40	54– 60
42	49– 58	75	35– 85	05	45– 69	43	47– 69
44	36– 79	76	48– 62	08	42– 74	45	55– 59
47	39– 73	82	42– 71	11	51– 61	48	56– 58
48	32– 89	87	29–103	15	35– 89	49	57– 57
49	37– 77	88	36– 83	16	41– 76	50	50– 65
50	50– 57	89	49– 61	20	52– 60	55	35– 93
52	46– 62	90	46– 65	24	44– 71	56	44– 74
56	51– 56	92	44– 68	27	53– 59	64	51– 64
60	52– 55	93	41– 73	28	46– 68	66	46– 71
62	53– 54	96	28–107	31	31–101	67	33– 99
67	47– 61	97	37– 81	32	54– 58	68	43– 76
70	41– 70	3000	50– 60	35	55– 57	70	30–109
71	33– 87	02	38– 79	36	56– 56	76	42– 78
80	45– 64	03	39– 77	39	43– 73	77	29–113
81	43– 67	07	31– 97	45	37– 85	80	41– 80
83	31– 93	08	47– 64	49	47– 67	83	49– 67
86	39– 74	09	51– 59	50	50– 63	85	45– 73
88	38– 76	10	43– 70	54	38– 83	86	53– 62
89	27–107	15	45– 67	57	41– 77	90	47– 70
90	34– 85	16	52– 58	59	39– 81	93	37– 89
91	49– 59	21	53– 57	60	40– 79	94	54– 61
98	46– 63	24	48– 63	61	29–109	98	34– 97
2900	50– 58	25	55– 55	62	51– 62	3300	50– 66
04	44– 66	26	34– 89	64	28–113	04	56– 59
05	35– 83	30	30–101	68	48– 66	06	57– 58
07	51– 57	34	41– 74	72	52– 61	11	43– 77
10	30– 97	36	46– 66	74	46– 69	12	48– 69
11	41– 71	38	49– 62	80	53– 60	15	51– 65
12	52– 56	40	40– 76	82	43– 74	17	31–107
14	47– 62	42	39– 78	85	49– 65	18	42– 79
15	53– 55	45	35– 87	86	54– 59	20	40– 83
16	54– 54	50	50– 61	90	55– 58	21	41– 81
20	40– 73	51	27–113	92	56– 57	25	35– 95
23	37– 79	52	28–109	93	31–103	28	52– 64
24	43– 68	53	43– 71	95	45– 71	30	45– 74
25	45– 65	55	47– 65	96	47– 68	32	49– 68
26	38– 77	60	51– 60	98	41– 78	33	33–101
28	48– 61	66	42– 73	3200	50– 64	37	47– 71
29	29–101	68	52– 59	01	33– 97	39	53– 63
37	33– 89	69	33– 93	04	36– 89	44	44– 76
38	26–113	71	37– 83	10	30–107	48	54– 62
40	49– 60	72	48– 64	12	44– 73	50	50– 67
43	27–109	74	53– 58	13	51– 63	54	43– 78
44	46– 64	75	41– 75	16	48– 67	55	55– 61
45	31– 95	78	54– 57	19	37– 87	58	46– 73
48	44– 67	80	55– 56	20	46– 70	60	56– 60
50	50– 59	81	39– 79	24	52– 62	62	41– 82
52	41– 72	82	46– 67	25	43– 75	63	57– 59

FACTOR TABLE.—(*Continued*) 3364 to 3936

Number	Factors	Number	Factors	Number	Factors	Number	Factors
3364	58– 58	3498	53– 66	3648	57– 64	3784	44– 86
66	51– 66	3500	50– 70	49	41– 89	92	48– 79
67	37– 91	03	31–113	50	50– 73	95	55– 69
75	45– 75	04	48– 73	52	44– 83	96	52– 73
79	31– 109	10	54– 65	54	58– 63	3800	50– 76
80	52– 65	15	37– 95	55	43– 85	07	47– 81
81	49– 69	19	51– 69	57	53– 69	08	56– 68
82	38– 89	20	55– 64	58	59– 62	11	37–103
84	47– 72	25	47– 75	60	60– 61	13	41– 93
88	44– 77	26	43– 82	63	37– 99	15	35–109
90	30–113	28	56– 63	66	47– 78	16	53– 72
92	53– 64	31	33–107	72	54– 68	18	46– 83
93	39– 87	34	57– 62	75	49– 75	19	57– 67
95	35– 97	35	35–101	80	46– 80	22	49– 78
97	43– 79	36	52– 68	85	55– 67	25	51– 75
99	33–103	38	58– 61	86	38– 97	27	43– 89
3400	50– 68	40	59– 60	89	31–119	28	44– 87
02	54 –63	42	46– 77	90	45– 82	34	54– 71
03	41– 83	49	39– 91	92	52– 71	35	59– 65
04	46– 74	50	50– 71	96	56– 66	38	38–101
08	48– 71	51	53– 67	98	43– 86	40	60– 64
10	55– 62	52	48– 74	3700	50– 74	42	34–113
16	56– 61	55	45– 79	05	57– 65	43	61– 63
17	51– 67	60	40– 89	06	34–109	44	62– 62
20	57– 60	64	54– 66	10	53– 70	48	52– 74
22	58– 59	67	41– 87	12	58– 64	50	55– 70
24	32–107	69	43– 83	13	47– 79	52	36–107
30	49– 70	70	51– 70	17	59– 63	54	47– 82
31	47– 73	72	38– 94	20	60– 62	61	39– 99
32	52– 66	75	55– 65	21	61– 61	64	56– 69
34	34–101	77	49– 73	23	51– 73	69	53– 73
40	43– 80	84	56– 64	24	49– 76	70	45– 86
41	37– 93	88	52– 69	26	54– 69	71	49– 79
44	41– 84	89	37– 97	29	33–113	72	44– 88
45	53– 65	91	57– 63	31	41– 91	76	57– 68
50	50– 69	96	58– 62	35	45– 83	80	40– 97
56	54– 64	97	33–109	37	37–101	86	58– 67
58	38– 91	99	59– 61	38	42– 89	88	54– 72
65	55– 63	3600	60– 60	40	55– 68	94	59– 66
68	51– 68	04	53– 68	41	43– 87	95	41– 95
71	39– 89	08	41– 88	44	52– 72	3900	50– 78
72	56– 62	10	38– 95	45	35–107	01	47– 83
76	44– 79	12	42– 86	50	50– 75	04	61– 64
77	57– 61	16	32– 113	52	56– 67	05	55– 71
78	47– 74	18	54– 67	60	47– 80	06	62– 63
79	49– 71	19	47– 77	62	57– 66	10	46– 85
80	58– 60	21	51– 71	63	53– 71	13	43– 91
81	59– 59	26	49– 74	70	58– 65	15	45– 87
83	43– 81	27	39– 93	72	46– 82	16	44– 89
84	52– 67	30	55– 66	73	49– 77	20	56– 70
35	41– 85	34	46– 79	74	51– 74	22	53– 74
86	42– 83	36	36–101	76	59– 64	24	36–109
88	32–109	38	34–107	80	60– 63	27	51– 77
92	36– 97	40	56– 65	82	61– 62	33	57– 69
96	46– 76	45	45– 81	83	39– 97	36	48– 82

FACTOR TABLE.—(*Continued*) 3939 to 4550

Number	Factors	Number	Factors	Number	Factors	Number	Factors
3939	39–101	4087	61– 67	4240	53– 80	4392	61– 72
42	54– 73	88	56– 73	42	42–101	99	53– 83
44	58– 68	89	47– 87	48	59– 72	4400	55– 80
48	47– 84	92	62– 66	50	50– 85	02	62– 71
50	50– 79	94	46– 89	51	39–109	03	37–119
52	52– 76	95	63– 65	56	56– 76	07	39–113
53	59– 67	96	64– 64	57	43– 99	08	58– 76
55	35–113	4100	50– 82	60	60– 71	10	63 –70
56	46– 86	04	57– 72	63	49– 87	16	64– 69
59	37–107	08	52– 79	64	52– 82	18	47– 94
60	60– 66	16	49– 84	66	54– 79	20	65– 68
65	61– 65	18	58– 71	68	44– 97	22	66– 67
68	62– 64	25	55– 75	70	61– 70	24	56– 79
69	63– 63	28	48– 86	72	48– 89	25	59– 75
75	53– 75	30	59– 70	75	57– 75	28	54– 82
76	56– 71	31	51– 81	77	47– 91	29	43–103
77	41– 97	34	53– 78	78	62– 69	37	51– 87
78	51– 78	36	47– 88	80	40–107	40	60– 74
84	48– 83	40	60– 69	84	63– 68	44	44–101
90	57– 70	41	41–101	88	64– 67	46	57– 78
95	47– 85	42	38–109	90	65– 66	50	50– 89
96	54– 74	44	56– 74	92	58– 74	52	53– 84
99	43– 93	48	61– 68	93	53– 81	53	61– 73
4000	50– 80	50	50– 83	94	38–113	55	55– 81
02	58– 69	54	62– 67	4300	50– 86	59	49– 91
04	52– 77	58	63– 66	07	59– 73	62	46– 97
05	45– 89	60	64– 65	12	56– 77	64	62– 72
12	59– 68	61	57– 73	16	52– 83	65	47– 95
15	55– 73	65	49– 85	20	60– 72	66	58– 77
17	39–103	71	43– 97	24	47– 92	69	41–109
18	49– 82	73	39–107	29	39–111	72	52– 86
20	60– 67	76	58– 72	31	61– 71	73	63– 71
26	61– 66	80	55– 76	32	57– 76	80	64– 70
28	53– 76	81	37–113	35	51– 85	82	54– 83
29	51– 79	82	51– 82	40	62– 70	84	59– 76
30	62– 65	83	47– 89	43	43–101	85	65– 69
32	63– 64	85	45– 93	45	55– 79	88	66– 68
33	37–109	86	46– 91	46	53– 82	89	67– 67
40	40–101	87	53– 79	47	63– 69	94	42–107
42	47– 86	89	59– 71	50	58– 75	4500	60– 75
47	57– 71	4200	56– 75	52	64– 68	03	57– 79
48	46– 88	09	61– 69	55	65– 67	05	53– 85
50	50– 81	12	54– 78	56	66– 66	08	49– 92
56	52– 78	14	49– 86	60	40–109	10	55– 82
59	41– 99	16	62– 68	61	49– 89	12	48– 94
60	58– 70	18	57– 74	65	45– 97	14	61– 74
66	38–107	21	63– 67	66	59– 74	20	40–113
67	49– 83	23	41–103	68	56– 78	24	58– 78
68	36–113	24	64– 66	70	46– 95	26	62– 73
70	55– 74	25	65– 65	71	47– 93	36	63– 72
71	59– 69	30	47– 90	74	54– 81	39	51– 89
74	42– 97	32	46– 92	80	60– 73	43	59– 77
80	60– 68	33	51– 83	86	51– 86	44	64– 71
81	53– 77	34	58– 73	87	41–107	45	45–101
85	43– 95	35	55– 77	89	57– 77	50	50– 91

Factor Table.—(*Continued*) 4551 to 520

Number	Factors	Number	Factors	Number	Factors	Number	Factor
4551	41-111	4708	44-107	4872	58-84	5040	70-72
54	66-69	12	62-76	75	65-75	41	71-71
56	67-68	17	53-89	76	53-92	44	52-97
57	49-93	20	59-80	79	41-119	46	58-87
58	53-86	25	63-75	80	61-80	47	49-103
59	47-97	30	55-86	84	66-74	49	51-99
60	60-76	31	57-83	88	52-94	50	50-101
63	39-117	32	52-91	91	67-73	56	64-79
65	55-83	36	64-74	95	55-89	60	55-92
75	61-75	40	60-79	96	68-72	63	61-83
76	52-88	43	51-93	97	59-83	70	65-78
78	42-109	45	65-73	98	62-79	73	57-89
82	58-79	46	42-113	99	69-71	74	59-86
88	62-74	47	47-101	4900	70-70	76	54-94
90	54-85	50	50-95	02	57-86	82	66-77
92	56-82	52	66-72	05	45-109	84	62-82
99	63-73	53	49-97	14	63-78	85	45-113
4600	50-92	56	58-82	20	60-82	88	53-96
01	43-107	57	67-71	22	46-107	92	67-76
02	59-78	58	61-78	28	64-77	96	56-91
06	49-94	60	68-70	29	53-93	5100	68-75
08	64-72	61	69-69	30	58-85	03	63-81
11	53-87	70	53-90	40	65-76	04	58-88
15	65-71	73	43-111	41	61-81	06	69-74
17	57-81	74	62-77	47	51-97	10	70-73
20	66-70	79	59-81	49	49-101	12	71-72
23	67-69	84	52-92	50	66-75	15	55-93
24	68-68	85	55-87	56	59-84	17	43-119
28	52-89	88	57-84	58	67-74	20	64-80
33	41-113	94	51-94	59	57-87	23	47-109
36	61-76	96	44-109	60	62-80	24	61-84
40	58-80	4800	64-75	64	68-73	30	57-90
41	51-91	02	49-98	68	69-72	33	59-87
44	54-86	06	54-89	70	70-71	35	65 -79
46	46-101	10	65-74	72	44-113	36	48-107
48	56-83	14	58-83	77	63-79	41	53-97
50	62-75	15	45-107	80	60-83	46	62-83
53	47-99	16	56-86	82	53-94	48	66-78
55	49-95	18	66-73	84	56-89	51	51-101
56	48-97	19	61-79	88	58-86	52	56-92
61	59-79	23	53-91	92	64-78	59	67-77
62	63-74	24	67-72	98	51-98	60	60-86
64	53-88	28	68-71	5000	50-100	62	58-89
72	64-73	30	69-70	02	61-82	66	63-82
74	57-82	36	62-78	05	65-77	68	68-76
75	55-85	38	59-82	14	46-109	70	55-94
80	60-78	40	55-88	15	59-85	75	69-75
86	66-71	41	47-103	16	66-76	80	70-74
87	43-109	45	57-85	22	62-81	83	71-73
90	67-70	48	48-101	25	67-75	84	72-72
92	68-69	50	50-97	29	47-107	85	61-85
97	61-77	51	63-77	31	43-117	92	59-88
98	58-81	59	43-113	32	68-74	94	53-98
4700	50-94	60	60-81	35	53-95	98	46-113
04	56-84	64	64-76	37	69-73	5200	65-80

Factor Table.—(*Continued*) 5202 to 5883

Number	Factors	Number	Factors	Number	Factors	Number	Factors
5202	51–102	5369	59– 91	5546	59– 94	5712	68– 84
08	62– 84	72	68– 79	48	73– 76		Out
14	66– 79	76	64– 84	50	74– 75	20	65– 88
17	47–111	82	69– 78	51	61– 91	23	59– 97
20	60– 87	90	70– 77	55	55–101	24	54–106
25	55– 95	94	62– 87	59	51–109	27	69– 83
26	67– 78	95	65– 83	61	67– 83	33	63– 91
29	63– 83	96	71– 76	62	54–103	34	61– 94
32	48–109	5400	72– 75	64	52–107	40	70– 82
36	68– 77	02	73– 74	65	53–105	42	66– 87
38	54– 97	06	53–102	68	64– 87	51	81– 71
43	49–107	08	52–104	76	68– 82	57	57–101
44	69– 76	12	66– 82	80	62– 90	60	72– 80
46	61– 86	15	57– 95	86	57– 98	62	67– 86
47	53– 99	18	63– 86	89	69– 81	63	51–113
48	82– 64	24	48–113	90	65– 86	66	62– 93
50	70– 75	27	67– 81	93	47–119	67	73– 79
51	59– 89	28	59– 92	5600	70– 80	68	56–103
52	52–101	29	61– 89	05	59– 95	72	74– 78
53	51–103	32	56– 97	07	63– 89	75	75– 77
54	71– 74	39	49–111	09	71– 79	76	76– 76
56	72– 73	40	68– 80	10	66– 85	77	53–109
64	56– 94	45	55– 99	12	61– 92	78	54–107
65	65– 81	50	50–109	16	72– 78	80	68– 85
70	62– 85	51	69– 79	18	53–106	82	59– 98
78	58– 91	53	Out	21	73– 77	85	65– 89
80	66– 80	54	54–101	24	74– 76	95	61– 95
92	63– 84	56	62– 88	25	75– 75	96	69– 84
93	67– 79	57	51–107	26	58– 97	5800	58–100
5300	53–100	59	53–103	28	67– 84	08	66– 88
01	57– 93	60	70– 78	32	64– 88	10	70– 83
04	68– 78	67	71– 77	40	60– 94	14	57–102
07	61– 87	72	72– 76	42	62– 91	20	60– 97
10	59– 90	75	73– 75	43	57– 99	22	71– 82
11	47–113	76	74– 74	44	68– 83	24	56–104
12	64– 83	78	66– 83	50	50–113	28	62– 94
13	69– 77	81	63– 87	55	65– 87	29	67– 87
20	70– 76	87	59– 93	56	56–101	30	55–106
25	71– 75	88	56– 98	58	69– 82	31	49–119
28	72– 74	90	61– 90	61	51–111	32	72– 81
29	73– 73	94	67– 82	64	59– 96	40	73– 80
30	65– 82	99	47–117	65	55–103	41	59– 99
32	62– 86	5500	55–100	70	70– 81	46	74– 79
35	55– 97	04	64– 86	71	53–107	48	68– 86
36	58– 92	08	68– 81	73	61– 93	50	75– 78
40	60– 89	10	58– 95	76	66– 86	52	76– 77
41	49–109	12	53–104	80	71– 80	56	61– 96
46	66– 81	18	62– 89	84	58– 98	58	58–101
50	50–107	20	69– 80	88	72– 79	59	63– 93
53	53–101	25	65– 85	94	73– 78	65	69– 85
55	63– 85	29	57– 97	95	67– 85	71	57–103
56	52–103	30	70– 79	96	64– 89	74	66– 89
58	57– 94	37	49–113	98	74– 77	76	52–113
60	67– 80	38	71– 78	5700	75– 76	80	70– 84
68	61– 88	44	72– 77	04	62– 92	83	53–111

FACTOR TABLE.—(*Continued*) 5885 to 6634

Number	Factors	Number	Factors	Number	Factors	Number	Factors
5885	55–107	6068	74– 82	6241	79– 79	6435	65– 99
86	54–109	69	51–119	48	71– 88	38	74– 87
88	64– 92	72	69– 88	54	59–106	40	70– 92
90	62– 95	75	75– 81	56	68– 92	41	57–113
93	71– 83	76	62– 98	62	62–101	48	62–104
96	67– 88	77	59–103	64	72– 87	50	75– 86
5900	59–100	80	76– 80	70	66– 95	60	76– 85
04	72– 82	83	77– 79	72	64– 98	61	71– 91
13	73– 81	84	78– 78	78	73– 86	64	64–101
15	65– 91	90	70– 87	79	69– 91	66	61–106
16	68– 87	97	67– 91	83	61–103	68	77– 84
17	61– 97	99	57–107	90	74– 85	74	78– 83
20	74– 80	6100	61–100	98	67– 94	78	79– 82
22	63– 94	02	54–113	6300	70– 90	80	72– 90
25	75– 79	04	56–109	05	65– 97	86	69– 94
28	76– 78	06	71– 86	07	53–119	89	63–103
29	77– 77	10	65– 94	08	76– 83	90	59–110
34	69– 86	11	63– 97	13	59–107	97	73– 89
36	56–106	20	72– 85	14	77– 82	99	67– 97
40	66– 90	32	73– 84	18	78– 81	6500	65–100
50	70– 85	36	59–104	19	71– 89	10	70– 93
52	64– 93	38	66– 93	20	79– 80	12	74– 88
59	59–101	41	69– 89	22	58–109	25	75– 87
63	67– 89	42	74– 83	24	68– 93	27	76–107
64	71– 84	44	64– 96	27	57–111	28	68– 96
74	58–103	48	58–106	28	56–113	32	71– 92
76	72– 83	50	75– 82	36	72– 88	34	66– 99
78	61– 98	56	76– 81	44	61–104	36	76– 86
80	65– 92	60	77– 80	48	69– 92	40	60–109
84	68– 88	61	61–101	51	73– 87	45	77– 85
85	63– 95	62	78– 79	60	60–106	49	59–111
86	73– 82	64	67– 92	63	63–101	52	78– 84
89	53–113	74	63– 98	64	74– 86	54	58–113
92	56–107	75	65– 95	65	67– 95	55	69– 95
94	74– 81	77	71– 87	70	70– 91	57	79– 83
95	55–109	80	60–103	72	59–108	60	80– 82
6000	75– 80	88	68– 91	75	75– 85	61	81– 81
03	69– 87	92	72– 86	80	58–110	65	65–101
04	76– 79	95	59–105	84	76– 84	66	67– 98
06	77– 78	6200	62–100	86	62–103	70	73– 90
14	62– 97	04	66– 94	90	71– 90	72	62–106
16	64– 94	05	73– 85	91	77– 83	80	70– 94
18	59–102	06	58–107	92	68– 94	86	74– 89
20	70– 86	08	64– 97	96	78– 82	88	61–108
30	67– 90	10	69– 90	99	81– 79	92	64–103
32	58–104	13	57–109	6400	80– 80	96	68– 97
35	71– 85	15	55–113	02	66– 97	6600	75– 88
39	61– 99	16	74– 84	05	61–105	03	71– 93
42	57–106	22	61–102	08	72– 89	12	76– 87
45	65– 93	25	75– 83	17	69– 93	15	63–105
48	72– 84	30	70– 89	20	60–107	22	77– 86
50	55–110	31	67– 93	24	73– 88	24	72– 92
52	68– 89	32	76– 82	26	63–102	30	78– 85
59	73– 83	37	77– 81	31	59–109	33	67– 99
60	60–101	40	78– 80	32	67– 96	34	62–107

FACTOR TABLE.—(*Continued*) 6636 to 7482

Number	Factors	Number	Factors	Number	Factors	Number	Factors
6636	79– 84	6831	69– 99	7040	80– 88	7272	72–101
40	80– 83	34	67–102	47	81– 87	75	75– 97
42	81– 82	40	76– 90	50	75– 94	76	68–107
43	73– 91	48	64–107	52	82– 86	80	80– 91
49	61–109	53	77– 89	55	83– 85	90	81– 90
50	70– 95	60	70– 98	56	84– 84	96	76– 96
56	64–104	62	73– 94	62	66–107	98	82– 89
60	74– 90	64	78– 88	70	70–101	7300	73–100
64	68– 98	67	63–109	80	59–120	03	67–109
66	66–101	68	68–101	81	73– 97	04	83– 88
67	59–113	73	79– 87	84	77– 92	08	84– 87
74	71– 94	80	80– 86	85	65–109	10	85– 86
75	75– 89	82	74– 93	98	78– 91	13	71–103
78	63–106	85	81– 85	7100	71–100	14	69–106
88	76– 88	87	71– 97	02	67–106	15	77– 95
93	69– 97	88	82– 84	04	74– 96	20	61–120
95	65–103	89	83– 83	07	69–103	26	74– 99
96	72– 93	90	65–106	10	79– 90	32	78– 94
99	77– 87	93	61–113	19	63–113	44	72–102
6700	67–100	6900	75– 92	20	80– 89	45	65–113
08	78– 86	01	67–103	25	75– 95	47	79– 93
10	61–110	03	59–117	28	81– 88	50	75– 98
15	79– 85	12	72– 96	34	82– 87	60	80– 92
16	73– 92	16	76– 91	37	61–117	70	67–110
20	80– 84	30	77– 90	38	83– 86	71	81– 91
23	81– 83	35	73– 95	40	84– 85	72	76– 97
24	82– 82	36	68–102	44	76– 94	73	73–101
32	68– 99	42	78– 89	50	65–110	80	82– 90
34	74– 91	52	79– 88	54	73– 98	83	69–107
41	63–107	55	65–107	61	77– 93	84	71–104
45	71– 95	56	74– 94	69	67–107	87	83– 89
50	75– 90	58	71– 98	71	71–101	92	84– 88
58	62–109	60	80– 87	76	78– 92	95	85– 87
60	65–104	66	81– 86	78	74– 97	96	86– 86
62	69– 98	68	67–104	89	79– 91	7400	74–100
64	76– 89	69	69–101	94	66–109	10	78– 95
67	67–101	70	82– 85	7200	80– 90	12	68–109
68	72– 94	72	83– 84	08	68–106	16	72–103
71	61–111	75	75– 93	09	81– 89	20	70–106
76	77– 88	76	64–109	10	70–103	25	75– 99
80	60–113	84	72– 97	16	82– 88	26	79– 94
83	57–119	92	76– 92	20	76– 95	37	67–111
84	64–106	96	66–106	21	83– 87	40	80– 93
86	78– 87	7000	70–100	24	84– 86	46	73–102
89	73– 93	04	68–103	25	85– 85	48	76– 98
90	70– 97	06	62–113	27	73– 99	52	81– 92
94	79– 86	07	77– 91	32	64–113	55	71–105
98	66–103	08	73– 96	36	67–108	58	66–113
6800	80– 85	20	78– 90	38	77– 94	62	82– 91
04	81– 84	21	59–119	45	69–105	69	77– 97
06	82– 83	29	71– 99	52	74– 98	70	83– 90
08	74– 92	30	74– 95	54	78– 93	74	74–101
16	71– 96	31	79– 89	59	61–119	76	84– 89
20	62–110	35	67–105	60	66–110	80	85– 88
25	75– 91	38	69–102	68	79– 92	82	86– 87

FACTOR TABLE.—(*Continued*) 7488 to 8536

Number	Factors	Number	Factors	Number	Factors	Number	Factors
7488	78–96	7739	71–109	7998	86–93	8262	81–102
90	70–107	40	86–90	8000	80–100	65	87–95
97	63–119	42	79–98	04	87–92	68	78–106
7500	75–100	43	87–89	08	88–91	72	88–94
05	79–95	44	88–88	10	89–90	77	89–93
19	73–103	52	76–102	19	81–99	80	90–92
20	80–94	60	80–97	23	71–113	81	91–91
21	69–109	70	74–105	25	75–107	82	82–101
24	76–99	76	81–96	30	73–110	84	76–109
26	71–106	77	77–101	34	78–103	95	79–105
33	81–93	90	82–95	36	82–98	8300	83–100
44	82–92	97	69–113	40	67–120	07	71–117
46	77–98	7800	78–100	51	83–97	16	84–99
48	74–102	02	83–94	56	76–106	20	80–104
53	83–91	10	71–110	58	79–102	30	85–98
60	84–90	11	73–107	64	84–96	42	86–97
65	85–89	12	84–93	66	74–109	43	81–103
66	78–97	20	85–92	75	85–95	46	78–107
68	86–88	21	79–99	80	80–101	52	87–96
69	87–87	26	86–91	84	86–94	60	88–95
71	67–113	28	76–103	85	77–105	62	74–113
75	75–101	30	87–90	91	87–93	64	82–102
84	79–96	32	88–89	96	88–92	66	89–94
90	69–110	40	80–98	99	89–91	70	90–93
92	73–104	44	74–106	8100	90–90	72	91–92
97	71–107	48	72–109	03	73–111	74	79–106
7600	80–95	54	77–102	12	78–104	83	83–101
14	81–94	57	81–97	18	82–99	93	77–109
22	74–103	72	82–96	32	76–107	8400	84–100
23	77–99	75	75–105	34	83–98	15	85–99
26	82–93	78	78–101	36	72–113	24	81–104
30	70–109	84	73–108	37	79–103	28	86–98
32	72–106	85	83–95	40	74–110	39	87–97
36	83–92	96	84–94	48	84–97	46	82–103
44	84–91	7900	79–100	60	85–96	48	88–96
50	85–90	04	76–104	62	77–106	49	71–119
54	86–89	05	85–93	70	86–95	53	79–107
56	87–88	10	70–113	75	75–109	55	89–95
63	79–97	12	86–92	78	87–94	60	90–94
65	73–105	17	87–91	81	81–101	63	91–93
68	71–108	18	74–107	84	88–93	64	92–92
76	76–101	20	88–90	88	89–92	66	83–102
80	80–96	21	89–89	90	90–91	70	77–110
84	68–113	31	77–103	8200	82–100	75	75–113
95	81–95	38	81–98	08	76–108	80	80–106
96	74–104	50	75–106	11	69–119	84	84–101
7700	77–100	54	82–97	16	79–104	8500	85–100
04	72–107	56	78–102	17	83–99	02	78–109
08	82–94	57	73–109	32	84–98	05	81–105
19	83–93	68	83–96	39	77–107	14	86–99
22	78–99	73	67–119	40	80–103	20	71–120
25	75–103	79	79–101	45	85–97	26	87–98
28	84–92	80	84–95	49	73–113	28	82–104
35	85–91	90	85–94	50	75–110	32	79–108
38	73–106	92	74–108	56	86–96	36	88–97

FACTOR TABLE.—*(Continued)* 8541 to 9960

Number	Factors	Number	Factors	Number	Factors	Number	Factors
8541	73–117	8832	92–96	9163	77–119	9555	91–105
44	89–96	35	93–95	67	89–103	68	92–104
47	77–111	36	94–94	80	90–102	70	87–110
49	83–103	40	85–104	91	91–101	79	93–103
50	90–95	56	82–108	9200	92–100	88	94–102
54	91–94	58	86–103	02	86–107	92	88–109
56	92–93	74	87–102	07	93–99	95	95–101
60	80–107	80	74–120	12	94–98	9600	96–100
68	84–102	81	83–107	15	95–97	03	97–99
80	78–110	88	88–101	16	96–96	04	98–98
85	85–101	8900	89–100	22	87–106	05	85–113
86	81–106	04	84–106	40	88–105	12	89–108
88	76–113	10	90–99	56	89–104	30	90–107
8600	86–100	18	91–98	65	85–109	39	81–119
10	82–105	24	92–97	66	82–113	46	91–106
11	79–109	25	85–105	70	90–103	60	92–105
13	87–99	27	79–113	82	91–102	72	93–104
24	88–98	28	93–96	92	92–101	80	88–110
32	83–104	30	94–95	9300	93–100	82	94–103
33	89–97	38	82–109	06	94–99	88	86–108
40	90–96	44	86–104	09	87–107	90	95–102
45	91–95	61	87–103	10	95–98	96	96–101
48	92–94	64	83–108	12	96–97	9700	97–100
49	93–93	76	88–102	28	88–106	01	89–109
52	84–103	88	84–107	45	89–105	02	98–99
67	81–107	89	89–101	50	85–110	18	86–113
70	85–102	91	81–111	60	90–104	20	90–108
86	86–101	9000	90–100	73	91–103	37	91–107
87	73–119	09	91–99	74	86–109	52	92–106
90	79–110	10	85–106	79	83–113	65	93–105
92	82–106	16	92–98	84	92–102	76	94–104
8700	87–100	20	82–110	93	93–101	85	95–103
01	77–113	21	93–97	96	87–108	90	89–110
12	88–99	24	94–96	9400	94–100	92	96–102
15	83–105	25	95–95	01	79–119	97	97–101
20	80–109	30	86–105	05	95–99	9800	98–100
22	89–98	40	80–113	08	96–98	01	99–99
30	90–97	47	83–109	09	97–97	10	90–109
36	91–96	48	87–104	16	88–107	28	91–108
40	92–95	64	88–103	34	89–106	31	87–113
42	93–94	72	84–108	50	90–105	40	82–120
48	81–108	78	89–102	60	86–110	44	92–107
55	85–103	90	90–101	64	91–104	58	93–106
60	73–120	95	85–107	76	92–103	70	94–105
69	79–111	9100	91–100	77	81–117	77	83–119
72	86–102	08	92–99	80	79–120	80	95–104
74	82–107	14	93–98	83	87–109	88	96–103
87	87–101	16	86–106	86	93–102	94	97–102
98	83–106	18	94–97	92	84–113	98	98–101
8800	88–100	20	95–96	94	94–101	9900	99–100
11	89–99	30	83–110	9500	95–100	19	91–109
14	78–113	35	87–105	04	96–99	36	92–108
20	90–98	52	88–104	06	97–98	44	88–113
27	91–97	53	81–113	23	89–107	51	93–107
29	81–109	56	84–109	40	90–106	60	83–120

FACTOR TABLE.—(*Continued*) 9964 to 14400

Number	Factors	Number	Factors	Number	Factors	Number	Factors
9964	94–106	10464	96–109	11124	103–108	12019	101–119
75	95–105	476	97–108	128	104–107	51	103–117
84	96–104	486	98–107	130	105–106	91	107–113
91	97–103	494	99–106	160	93–120	99	109–111
96	98–102	500	100–105	187	99–113	100	110–110
99	99–101	504	101–104	211	101–111	120	101–120
10000	100–100	506	102–103	220	102–110	204	108–113
10	91–110	509	93–113	227	103–109	240	102–120
44	93–108	560	96–110	232	104–108	257	103–119
57	89–113	573	97–109	235	105–107	317	109–113
58	94–107	584	98–108	236	106–106	360	103–120
70	95–106	593	99–107	280	94–120	430	110–113
80	96–105	600	100–106	300	100–113	480	104–120
88	97–104	605	101–105	312	101–112	519	107–117
94	98–103	608	102–104	322	102–111	543	111–113
98	99–102	609	103–103	330	103–110	600	105–120
100	100–101	622	94–113	336	104–109	656	112–113
101	91–111	670	97–110	340	105–108	720	106–120
120	92–110	680	89–120	342	106–107	733	107–119
137	93–109	682	98–109	400	95–120	753	109–117
152	94–108	692	99–108	413	101–113	769	113–113
165	95–107	700	100–107	433	103–111	840	107–120
170	90–113	706	101–106	440	104–110	882	113–114
176	96–106	710	102–105	445	105–109	960	108–120
179	87–117	712	103–104	448	106–108	971	109–119
185	97–105	735	95–113	449	107–107	987	111–117
192	98–104	780	98–110	514	101–114	995	113–115
197	99–103	791	99–109	520	96–120	13080	109–120
200	100–102	800	100–108	526	102–113	108	113–116
201	101–101	807	101–107	543	97–119	200	110–120
230	93–110	812	102–106	550	105–110	209	111–119
246	94–109	815	103–105	554	106–109	221	113–117
260	95–108	816	104–104	556	107–108	334	113–118
272	96–107	829	91–119	583	99–117	447	113–119
282	97–106	848	96–113	615	101–115	560	113–120
283	91–113	881	93–117	639	103–113	923	117–119
290	98–105	890	99–110	640	97–120	14161	119–119
296	99–104	900	100–109	660	106–110	400	120–120
300	100–103	908	101–108	663	107–109		
302	101–102	914	102–107	664	108–108		
320	86–120	918	103–106	716	101–116		
323	93–111	920	104–105	752	104–113		
340	94–110	961	97–113	760	98–120		
353	87–119	989	99–111	770	107–110		
355	95–109	11000	100–110	772	108–109		
368	96–108	09	101–109	781	99–119		
379	97–107	16	102–108	817	101–117		
388	98–106	21	103–107	865	105–113		
395	99–105	24	104–106	877	107–111		
396	92–113	25	105–105	880	108–110		
400	100–104	40	92–120	881	109–109		
403	101–103	67	93–119	918	101–118		
404	102–102	074	98–113	978	106–113		
440	87–120	110	101–110	990	109–110		
450	95–110	118	109–102	12000	100–120		

ANSWERS

Page	*Problem*	*Symbol*	*Variable*	*Answer*
152	1	*A* & *W*	.750 & .405	Lead
	2	*A* & *W*	.750 & .405	1.5
	3	*A* & *W*	.750 & .405	.38624
	4	*A* & *W*	.750 & .405	.07290
	5	*A* & *W*	.750 & .405	13°16′56″
	6	*F* & *W*	3 & .225	.16838
	7	*F* & *W*	3 & .225	1.9198
153	8	*P* & *W*	.750 & .600	1.3333
	9	*P* & *W*	.750 & .600	.43301
	10	*P* & *W*	.750 & .600	.25050
	11	*H* & *W*	9 & .065	.22222
	12	*H* & *W*	9 & .065	3°4′6″
	13	*H* & *W*	9 & .065	.02732
	14	*J*	.904	.42520
	15	*J*	.904	19°49′42″
161	1	*G*	34	.16666
	2	*G*	34	6.0000
	3	*H*	1.25	.07847
	4	*H*	1.25	.05147
	5	*H*	1.25	20
162	6	*S*	8	8.5625
	7	*S*	8	.01962
	8	*J*	22	.30814
	9	*J*	22	3.1428
	10	*J*	22	3.4285
	11	*J*	22	10.285
	12	*J*	22	2
163	13	*A*	2.5	.15703
	14	*A*	2.5	.15688
	15	*A*	2.5	3.2000
	16	*A*	2.5	3.3000
	17	*A*	2.5	1.9000
	18	*A*	2.5	10
	19	*A*	2.5	.10176
	20	*A*	2.5	2.1000
	21	*A*	2.5	4.7000
	22	*A* & *T*	2.5 & 4.75	29°8′26″

Page	Problem	Symbol	Variable	Answer
163	23	A & T	2.5 & 4.75	1.5582
164	24	U	3.9545	11
	25	U	3.9545	9.6363
	26	U	3.9545	1.9090
165	27	R	3.408	.39270
	28	R	3.408	5.8750
	29	R	3.408	.59550
	30	R	3.408	.72050
167	1	P	$\frac{4}{5}$	6.2500
	2	P	$\frac{4}{5}$	4.1500
	3	P	$\frac{4}{5}$	5.0000
	4	P	$\frac{4}{5}$	.43140
	5	P	$\frac{4}{5}$	.39245
	6	P	$\frac{4}{5}$	.21100
170	1	P'	.1875	2.2081
	2	P'	.1875	1.4920
171	3	P'	.1875	.09371
	4	P'	.1875	.06143
176	1	A	.135	.83492
	2	A	.135	.65920
177	3	A	.103	.33875
	4	A	.103	.15074
180	1	G	4	.41952
	2	G	4	2.0237
	3	G	4	.43756
	4	G	4	2.0550
181	5	H	6	.26578
	6	H	6	1.8542
	7	H	6	.27174
	8	H	6	1.8440
192	1	n	20	23°57′44″
	2	n	20	2°19′32″
	3	n	20	4.1036
	4	n	20	3.3333
	5	n	20	1.7355
	6	n	20	.81782
	7	n	20	7.6355
	8	n	20	3.6378
	9	n	20	.27555
	10	n	20	.14784
	11	N	52	1.2410
	12	N	52	1.0541
	13	N	52	5.9131
	14	N	52	4.6205

Page	Problem	Symbol	Variable	Answer
192	15	N	52	1.7762
	16	N	52	1.2148
	17	N	52	52°25′52″
	18	N	52	.23966
	19	N	52	54°10′37″
	20	N	52	35°32′54″
193	21	N	32	1.3888
	22	N	32	1.5138
	23	N	32	5.3333
	24	N	32	5.5689
	25	N	32	5.5689
	26	N	32	2.0248
	27	N	32	2.1056
	28	N	32	1.2570
	29	N	32	47°31′50″
	30	N	32	42°4′23″
195	1	N	28	33°16′32″
	2	N	28	38°28′4″
	3	N	28	2.8654
	4	N	28	.29415
	5	N	28	7.4042
	6	N	28	2.2781
	7	N	34	25°14′28″
	8	N	34	79°14′49″
198	1	γ	78°	5.8172
	2	γ	78°	5.6030
	3	γ	125°	1.1445
	4	γ	125°	3.0169
200	1*a*	N	50	68°34′33″
	1*b*	N	50	20°4′00″
206	1	N	43	1.8908
	2	N	43	2.6391
	3	N	43	9.1682
	4	N	43	8.2116
	5	N	43	8.5935
	6	N	43	.41196
	7	N	43	4.1634
	8	N	43	4.5453
	9	N	43	.19098
	10	N	43	.18600
	11	N	43	2°37′36″
	12	N	60	5.9681
207	13	N	60	3.9840
	14	N	60	.90050

Page	Problem	Symbol	Variable	Answer
207	15	*N*	60	6.4083
	16	*N*	60	2°50′50″
	17	*N*	53	6.5647
	18	*N*	53	3.3307
	19	*N*	53	4°24′55″
	20	*N*	47	.96585
	21	*N*	47	7.0942
	22	*N*	47	10°43′40″
213	1	*T*	8	3.6340
	2	*T*	8	3.8840
	3	*T*	8	$3\frac{1}{2}$
	4	*T*	8	40.841
	5	*T*	8	.125
	6	*T*	8	.26962
	7	*T*	8	.19622
	8	*T*	8	.12796
	9	*T*	8	.40774
	10	*S*	3.545	3.3677
	11	*S*	3.545	3.7222
	12	*S*	3.545	19°54′00″
	13	*S*	3.545	4.0554
	14	*S*	3.545	3.7010
220	1	*U*	5.0659	47°5′00″
	2	*U*	5.0659	6.8275
221	3	*T*	4.1272	41°7′00″
	4	*T*	4.1272	2.2810
	5	*S*	8.3986	47°6′00″
	6	*S*	8.3986	10.921
	7	*S*	8.3986	.22425
	8	*S*	8.3986	.14537
	9	*S*	8.3986	6.1615
	10	*S*	8.3986	11.206
222	11	*J*	2.7468	21°55′
	12	*J*	2.7468	28°5′
	13	*J*	2.7468	3.3684
	14	*J*	2.7468	2.1251
	15	*J*	2.7468	3.6184
	16	*J*	2.7468	2.3751
	17	*J*	2.7468	.19622
	18	*J*	2.7468	.12796
	19	*K*	5.5047	16°16′27″
	20	*K*	5.5047	76°16′27″
	21	*K*	5.5047	4.6876
	22	*K*	5.5047	6.3217

Page	Problem	Symbol	Variable	Answer
222	23	*K*	5.5047	50.444
	24	*K*	5.5047	4.8508
234	1	*U*	4.3500	11
	2	*U*	4.3500	2.4818
	3	*U*	4.3500	15.767
	4	*U*	4.3500	24°37′10″
	5	*U*	4.0000	15
	6	*U*	4.0000	71.979
	7	*U*	4.0000	14°50′8″
	8	*U*	4.0000	2.0643
236	1	*F*	1.437	10°31′6″
	2	*F*	1.437	40°44′55″
	3	*F*	1.437	30°11′3″
242	1	*G*	14.926	880.63
	2	*H*	9.25	8.6046
	3	*J*	20.5	42.983
243	4	*T*	5.75	178.25
	5	*S*	4.125	28.875
	6	*r*	.75	.17617
244	7	*U*	9.86	14.312
	8	*L*	20.3	.54133
248	1	*G*	1.253	7.5857
	2	*J*	9.5	3.3529
	3	*J*	9.5	2
249	4	*H*	3.75	7.0311
	5	*T*	11.5	1837.1
	6	*R*	23.75	.14866
250	7	*S*	77	55
	8	*N*	.625	111.01
	9	*P*	2.12	2.4352
	10	*L*	8.5	7.3995
254	1	*R*	465	3 × 5 × 31
	2	*S*	423	3 × 3 × 47
	3	*T*	665	5 × 7 × 19
	4	*J*	5293	67 × 79
	5	*U*	6351	73 × 87
	6	*N*	4144	56 × 74
	7	*M*	5226	67 × 78
	8	*L*	3526	43 × 82
	9	*E*	1343	17 × 79
	10	*P*	1998	37 × 54
	11	*W*	2511	31 × 81
	12	*F*	2744	49 × 56

Page	Problem	Symbol	Variable	Answer
255	13	H	841	$\frac{AC}{BD} = \frac{31 \times 41}{29 \times 29}$
	14	G	2482	$\frac{AC}{BD} = \frac{37 \times 97}{34 \times 73}$
	15	K	2891	$\frac{AC}{BD} = \frac{59 \times 71}{49 \times 59}$
	16	V	3198	$\frac{AC}{BD} = \frac{61 \times 67}{41 \times 78}$
	17	Q	6557	$\frac{AC}{BD} = \frac{73 \times 97}{79 \times 83}$
	18	Z	5893	$\frac{AC}{BD} = \frac{83 \times 97}{71 \times 83}$
258	1	N	391	$\frac{AC}{BD} = \frac{19 \times 23}{17 \times 23}$
	2	M	288	$\frac{AC}{BD} = \frac{17 \times 21}{16 \times 18}$
	3	L	750	$\frac{AC}{BD} = \frac{29 \times 31}{25 \times 30}$
	4	E	1989	$\frac{AC}{BD} = \frac{47 \times 57}{39 \times 51}$
	5	P	36	$\frac{AC}{BD} = \frac{5 \times 7}{4 \times 9}$
	6	W	4956	$\frac{AC}{BD} = \frac{57 \times 93}{59 \times 84}$
	7	K	7.15	$\frac{AC}{BD} = \frac{23 \times 49}{13 \times 55}$
	8	F	2.16	$\frac{AC}{BD} = \frac{12 \times 18}{15 \times 20}$
262	1	None	None	2°30′
265	7	None	None	118°7′30″
	7	None	None	6.8618
268	13	None	None	90°
	13	None	None	2.8284
269	19	None	None	117°55′52″
	19	None	None	3.1061
271	25	None	None	117°45′00″
	25	None	None	3.2101
272	31	None	None	.0000149 rev.
	31	None	None	opposite
275	1	E	$\frac{3}{4}$	30
	2	F	9.5	380
	3	G	$\frac{9}{13}$	27.692
	4	H	$\frac{1}{51}$	.78431
	5	J	17	17

Page	Problem	Symbol	Variable	Answer
275	6	K	28	21 or 49
	7	L	38	19
	8	M	48	15
	9	N	92	10
	10	P	65	24

The following four (4) problems have many answers; all of which may be correct. One answer is given.

278	1	R	141	$\frac{A}{B} = \frac{6}{21}$ $\frac{D}{E} = \frac{20}{70}$
	2	S	367	$\frac{A}{B} = \frac{3}{21}$ $\frac{D}{E} = \frac{21}{27}$
	3	T	519	$\frac{A}{B} = \frac{2}{27}$ $\frac{D}{E} = \frac{42}{27}$
	4	U	1997	$\frac{A}{B} = \frac{1}{49}$ $\frac{D}{E} = \frac{37}{49}$
286	1	D	$\frac{1}{5}$	3
	2	E	$\frac{7}{9}$	16
287	3	R	$\frac{5}{8}$	9
	4	G	$\frac{7}{54}$	$\frac{32}{247}$
	5	H	$\frac{8}{13}$	5
	6	K	$\frac{55}{87}$	.00033
	7	S	$\frac{1}{2}$	6
	8	M	$\frac{6}{19}$	$\frac{27}{85}$
	9	P	$\frac{23}{70}$	79
	10	T	$\frac{5}{7}$	3
294	1	X	K	$\frac{8307}{18385}$
	2	D	11	$\frac{63}{19}$
	3	E	5476	$\frac{87}{65}$
295	4	G	.4345	$\frac{10}{23}$
	5	X	N	$\frac{27949}{36045}$
	6	X	N	.77539
	7	H	371.525	$\frac{151}{561}$
	8	J	1.1654	$\frac{613}{526}$
	9	K	.41692	$\frac{823}{1974}$
	10	X	E	$\frac{37}{81}$

The following problems have many answers, all of which may be close enough for practical purposes. The answers given here have the least variation.

305	1	S	17.2463	$\frac{AC}{BD} = \frac{77 \times 93}{41 \times 100}$
	2	S	17.2463	$\frac{AC}{BD} = \frac{37 \times 38}{23 \times 35}$

Page	Problem	Symbol	Variable	Answer
305	3	R	1.49675	$\frac{AC}{BD} = \frac{56 \times 70}{27 \times 97}$
	4	R	1.49675	$\frac{AC}{BD} = \frac{71 \times 94}{49 \times 91}$
	5	M	7.2469	$\frac{AC}{BD} = \frac{41 \times 85}{29 \times 69}$
306	6	M	7.2469	$\frac{AC}{BD} = \frac{59 \times 89}{45 \times 67}$
	7	H	9.965	$\frac{AC}{BD} = \frac{61 \times 71}{67 \times 98}$
	7	H	9.965	$\frac{AC}{BD} = \frac{29 \times 29}{25 \times 51}$
	8	J	3.678	$\frac{AC}{BD} = \frac{35 \times 37}{27 \times 57}$
	8	J	3.678	$\frac{AC}{BD} = \frac{13 \times 89}{25 \times 55}$

The answers given here produce a lead slightly greater than the original lead with the least variation.

Page	Problem	Symbol	Variable	Answer
308	1	L	.5134	$\frac{AC}{BD} = \frac{47 \times 97}{30 \times 37}$
	2	K	.9241	$\frac{AC}{BD} = \frac{68 \times 53}{26 \times 25}$
	3	M	.4417	$\frac{AC}{BD} = \frac{28 \times 46}{27 \times 27}$
	4	N	.7513	$\frac{AC}{BD} = \frac{76 \times 76}{31 \times 31}$
	5	P	.5326	$\frac{AC}{BD} = \frac{48 \times 97}{31 \times 47}$
	6	Q	.2638	$\frac{AC}{BD} = \frac{38 \times 83}{49 \times 61}$
314	1	T	3.786	$\frac{AC}{BD} = \frac{83 \times 92}{49 \times 59}$
	2	M	80.768	$\frac{AC}{BD} = \frac{1 \times 13}{7 \times 15}$
	3	L	56.431	$\frac{AC}{BD} = \frac{23 \times 24}{35 \times 89}$
	4	N	57.618	$\frac{AC}{BD} = \frac{13 \times 27}{32 \times 79}$
	5	P	24.972	$\frac{AC}{BD} = \frac{17 \times 61}{39 \times 83}$
	6	Q	35.133	$\frac{AC}{BD} = \frac{10 \times 12}{17 \times 31}$

Page	Problem	Symbol	Variable	Answer
317	1	N	9	$\frac{AC}{BD} = \frac{55 \times 55}{57 \times 78}$
	2	N	9	$\frac{AC}{BD} = \frac{4 \times 8}{3 \times 9}$
	3	N	9	$\frac{AC}{BD} = \frac{16 \times 8}{9 \times 9}$
	4	N	4	$\frac{AC}{BD} = \frac{5 \times 7}{3 \times 6}$
	5	N	4	$\frac{AC}{BD} = \frac{11 \times 11}{6 \times 6}$

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